Master's thesis in mathematics

Riemannian Submersions of Euclidean Space

Submitted by
Marcel Wunderlich

Supervised by
Prof. Dr. Burkhard Wilking

Westfälische Wilhelms-Universität Münster

December 18, 2014

Contents

Preface

This thesis is divided into three parts. In the first section we introduce the reader to Riemannian submersions and foliations, and state some general results. The second section deals with the titular topic, namely Riemannian submersions of Euclidean space. We first show that in this situation there is one totally geodesic fiber, and then proceed to show how this can be used to establish that these submersions arise from Lie group actions, if the codimension of the fibers is ≤ 3. The third and fourth section supplement this. In the third section we prove a small theorem, stating that any embedded submanifold in Euclidean space with flat normal bundle locally defines a foliation. The fourth section gives a short overview over generalizations of the main theorem and possible approaches in higher codimensions.

1 Preliminaries

In this section we introduce the vocabulary we need to talk about Riemannian submersions and prove some basic results which we will need throughout this thesis. There are several ways to introduce these concepts; we develop them starting from submersions. Some of these result are based on [O'N66] of O'Neill and we do not give a proof for these, but refer the reader to this beautiful work, instead.

1.1 Submersions

Let M and B smooth manifolds. A smooth map $\pi : M \to B$ is called a *submersion*, if π_* has maximal rank.

Definition 1.1 (Riemannian submersions)**.** Given a submersion $\pi : M \to B$, $\ker \pi_* =: \mathcal{V}$ defines a *distribution on* M, i.e. an smooth subbundle of TM. We call it the *vertical distribution* and vectors tangent to fibers *vertical.* In case that M is a Riemannian manifold we have a canonical orthogonal splitting of $TM = \mathcal{V} \oplus \mathcal{H}$ w.r.t. the metric, where $\mathcal{H}$ is called the *horizontal distribution*, and vector fields $X \in \Gamma\mathcal{H}$ are called *horizontal.* For each vector field X_* on B we have a unique horizontal lift X on M such that X and X_* are π-related. Such an X is called *basic.* A 1-form is called *basic*, if its corresponding metric dual vector field is basic.

If both M and B are Riemannian manifolds, π is called a *Riemannian submersion*, if π is a submersion and an isometry, when restricted to horizontal vectors.

Let us first fix some notations. We use $\mathcal{V}M$ as a shorthand for $\Gamma(TM)$, i.e. the space of vector fields of M. This notation should not lead to confusion with the vertical bundle $\mathcal{V}$, since for the latter we omit the manifold M.

Throughout this thesis, for $E \in \mathcal{V}M$ we will write E^h and E^v for the corresponding projections on the horizontal and vertical space. Projections of the covariant derivative $\nabla_E F$ will be abbreviated by $\nabla^h_E F$ and $\nabla^v_E F$ respectively. Usually we will write the scalar product notation $\langle , \rangle$ for Riemannian metrics. The normal bundle of a submanifold $N \subset M$ will be denoted by νN.

For Riemannian submersions, note the following crucial identities.

Lemma 1.2. Let $\pi : M \to B$ be a Riemannian submersion, X, Y be basic vector fields on M, $X_* := \pi_* X, Y_* := \pi_* Y$ the π-related vector fields on B. Then:

(i) $\langle X, Y\rangle = \langle X_*, Y_*\rangle \circ \pi$

(ii) $[X, Y]^h$ is the basic vector field corresponding to $[X_*, Y_*]$

(iii) $\nabla^h_X Y := (\nabla_X Y)^h$ is the basic vector field corresponding to $\nabla_{X_*} Y_*$

Proof. [O'N66, Lemma 1] □

Morally Riemannian submersions are the dual of isometric immersion, since the latter generalize isometries $M \to B$ for the case that $\dim M \leq \dim B$, and the former for the case that $\dim M \geq \dim B$. O'Neill established in [O'N66] that, akin to the way the second fundamental tensor controls an isometric immersion, there are two tensors which control a Riemannian submersion.

Definition 1.3 (The A- and S-tensors)**.** The *A-tensor* and *S-tensor* of a Riemannian submersion are given by

$$A_E F = (\nabla_{E^h} F^h)^v \qquad\qquad S_E F = -(\nabla_{F^v} E^h)^v,$$

for $E, F \in \mathcal{V}M$. Note that S is just the second fundamental tensor of the fibers and is self-adjoint. Furthermore, let A^* denote the pointwise adjoint of A. In fact, $A^*_X U$ is given by $-\nabla^h_X U$; for this let X, Y be horizontal, U be vertical. Then

$$\langle A^*_X U, Y\rangle = X\langle Y, U\rangle - \langle Y, \nabla_X U\rangle = \langle -\nabla^h_X U, Y\rangle.$$

Remark 1.4. This definition differs from the *O'Neill-tensors* A' and T originally given in [O'N66], namely

$$T_E F = (\nabla_{E^v} F^v)^h + (\nabla_{E^v} F^h)^v$$

and

$$A'_E F = (\nabla_{E^h} F^h)^v + (\nabla_{E^h} F^v)^h.$$

However note that restricted to horizontal fields, A' and A coincide.

Note that A and S are tensorial in both arguments: Clearly A is tensorial in the first argument, and S in the second, since the covariant derivative is tensorial in the first argument. Let $\varphi : M \to \mathbb{R}$ be a smooth function. Then

$$A_E \varphi F = \varphi \nabla^v_{E^h} F^h + (E^h(\varphi) F^h)^v = \varphi A_E F,$$

and similarly

$$S_{\varphi E} F = -\varphi \nabla^v_{F^v} E^h - (F^v(\varphi) E^h)^v = \varphi S_E F.$$

Since vertical vector fields are tangent to the fibers, which are embedded submanifolds, their Lie bracket is again vertical. However, this is also true for the bracket of a vertical field and one, that is π-related to a vector field downstairs, as the following lemma shows.

Lemma 1.5. If $\pi : M \to B$ is a submersion, then for an X π-related to a vector field X_* on B and vertical U, the Lie bracket $[X, U]$ is vertical.

Proof. This immediately follows from U being π-related to the zero vector field and X being π-related to X_*. Then the corresponding Lie brackets are also π-related, hence

$$\pi_*[X, U] = [\pi_* X, \pi_* U] = [X_*, 0] = 0,$$

and it follows that $[X, U]$ is vertical. □

In particular, this lemma applies in the situation of a Riemannian submersion to vertical U and basic X and then says that $[U, X]^h = 0$, a fact we will often use.

Lemma 1.6. The restriction of A to horizontal vectors x, y is simply the integrability tensor $\frac{1}{2}[X, Y]^v$, where X and Y are any extensions of x and y.

Proof. Note that by the same reasoning used to establish that A and S are tensorial, also $[X, Y]^v$ is tensorial in both arguments. So we may assume that X and Y are basic. Then $[V, X]$ and $[V, Y]$ are vertical, hence

$$\begin{aligned} 0 = V\langle X, Y\rangle &= \langle \nabla_V X, Y\rangle + \langle X, \nabla_V Y\rangle \\ &= \langle \nabla_X V + [V, X], Y\rangle + \langle X, \nabla_Y V + [V, Y]\rangle \\ &= \langle -A^*_X V, Y\rangle + \langle X, -A^*_Y V\rangle = \langle A_X Y, V\rangle + \langle A_Y X, V\rangle. \end{aligned}$$

Hence $A_X Y = -A_Y X$. But we also have $A_X Y - A_Y X = [X, Y]^v$, so

$$[X, Y]^v = A_X Y - A_Y X = 2A_X Y.$$

□

Remark 1.7. Furthermore T and S are related via

$$T_U V = \sum_i \langle S_{X_i} U, V \rangle X_i$$

where the X_i form a local orthonormal base of the horizontal distribution and U, V are vertical.

As mentioned, these tensors control the submersion. While we do not use this result, the following theorem is still noteworthy.

Theorem 1.8 (O'Neill, 1966)**.** Let π, $\bar{\pi}$ be Riemannian submersions of a connected Riemannian manifold M onto B. If π and $\bar{\pi}$ have the same tensors A' and T, and if their derivatives agree at one point in M, then $\pi = \bar{\pi}$.

Proof. [O'N66, Theorem 4] □

Lemma 1.9 (Criterion for a submersion being Riemannian)**.** Let $\pi : M \to B$ be a smooth submersion and M a Riemannian manifold. Then there exists a metric on B for which π becomes Riemannian iff $\mathcal{L}_U g^h$ vanishes in any vertical direction U, where g^h denotes the restriction of the metric of M to horizontal fields.

Proof. First assume

$$(\mathcal{L}_U g^h)(X, Y) = U\langle X, Y \rangle - \langle [U, X]^h, Y \rangle - \langle X, [U, Y]^h \rangle = 0.$$

Let $X_*, Y_* \in \mathcal{V}B$ and X, Y be their corresponding basic lifts. Then $[U, Y]$ and $[U, X]$ are vertical, so it follows that $U\langle X, Y \rangle = 0$. Hence we can define a metric h on B via $h(X_*, Y_*) := \langle X, Y \rangle$ and then clearly $\pi : (M, g) \to (B, h)$ is Riemannian.

Conversely, suppose π is a Riemannian submersion, let U be any vertical vector field and E_i a local orthonormal basic basis of the horizontal distribution. Then any horizontal vector fields X, Y have the form $X = \varphi^i E_i$, $Y = \psi^i E_i$, where we employ the Einstein summation convention. Then

$$\begin{aligned} \nabla_U X &= \nabla_U \varphi^i E_i = \varphi^i \nabla_U E_i + U(\varphi^i) E_i \\ [U, X] &= [U, \varphi^i E_i] = \varphi^i [U, E_i] + U(\varphi^i) E_i, \end{aligned}$$

and similar formulas hold for $\nabla_U Y$ and $[U, Y]$. It follows that

$$\begin{aligned} &(\mathcal{L}_U g^h)(X, Y) = U\langle X, Y \rangle - \langle [U, X], Y \rangle - \langle X, [U, Y] \rangle \\ &= \langle \nabla_U \varphi^i E_i, Y \rangle + \langle X, \nabla_U \psi^i E_i \rangle - \langle [U, \varphi^i E_i], Y \rangle - \langle X, [U, \psi^i E_i] \rangle \\ &= \varphi^i \langle \nabla_U E_i, Y \rangle + \langle U(\varphi^i) E_i, Y \rangle + \psi^i \langle X, \nabla_U E_i \rangle + \langle X, U(\psi^i) E_i \rangle \\ &- \varphi^i \langle [U, E_i], Y \rangle - \langle U(\varphi^i) E_i, Y \rangle - \psi^i \langle X, [U, E_i] \rangle - \langle X, U(\psi^i) E_i \rangle. \end{aligned}$$

Clearly the terms containing the derivatives of the φ^i and ψ^i cancel each other out. By Lemma 1.5 the Lie bracket of a vertical and a basic vector field is vertical, so $\varphi^i\langle[X, E_i], Y\rangle = 0$ and $\psi^i\langle X, [U, E_i]\rangle$. Therefore the calculation simplifies to

$$\begin{aligned}(\mathcal{L}_U g^h)(X, Y) &= \varphi^i\langle\nabla_U E_i, Y\rangle + \psi^i\langle X, \nabla_U E_i\rangle \\ &= \varphi^i\psi^j\langle\nabla_U E_i, E_j\rangle + \varphi^j\psi^i\langle E_j, \nabla_U E_i\rangle. \\ &= \varphi^i\psi^j U\langle E_i, E_j\rangle = 0.\end{aligned}$$

□

Corollary 1.10. A submersion $\pi : M \to B$ is Riemannian if and only if $\nabla^v : \mathcal{H} \times \mathcal{H} \to \mathcal{V}$ is skew-symmetric.

Proof. As ∇^v is tensorial, we may do this calculation using basic vector fields X, Y. Let U be any vertical field. By the preceding lemma π is Riemannian iff $\mathcal{L}_U g^h$ vanishes, and since X, Y are basic, $[U, X]$ and $[U, Y]$ are vertical, hence π is Riemannian iff $0 = (\mathcal{L}_U g^h)(X, Y) = U\langle X, Y\rangle$. Now the result follows from the calculations in the proof of Lemma 1.6. □

One fundamental question discussed in [O'N66] is how one can relate the curvature of M to the one of B. For sectional curvatures, O'Neill established the following very useful result, which is referred to as *O'Neill's formula.*

Theorem 1.11 (O'Neill's Formula, 1966)**.** Let $\pi : M \to B$ be a Riemannian submersion, K denote the sectional curvature of M, K_* the sectional curvature of B. Then, for horizontal vectors x and y on M

$$K(\sigma_{x,y}) = K_*(\sigma_{\pi_* x, \pi_* y}) - 3\frac{\|A_x y\|^2}{\|x \wedge y\|^2},$$

where $\sigma_{x,y}$ denotes the plane spanned by x and y.

Proof. [O'N66, Corollary 1] □

We conclude this section with a motivating example and the definition of homogeneity.

Example 1.12 (Glide rotations)**.** Consider the orbit fibration $\pi : \mathbb{R}^3 \to B^2 = \mathbb{R}^3/\mathbb{R}$, where $\mathbb{R}$ is the Lie group of isometries acting on $\mathbb{C} \times \mathbb{R}$ via glide-rotations, i.e.

$$t.(z, r) = (e^{it}z, r + t).$$

Then π is a Riemannian submersion and the z-axis is the only totally geodesic fiber. See Figure 1.

More generally we can look at any isometric Lie group action of $\mathbb{R}^k$ on a manifold M^n. Then the projection map $\pi : M^n \to M^n/\mathbb{R}^k$ becomes a Riemannian submersion. This motivates the following definition.

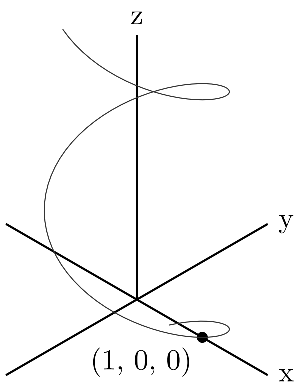

Figure 1: A fiber of the screwing motion through the point $(1,0,0)$. Note that the z-axis is a totally geodesic fiber.

Definition 1.13 (Homogeneity)**.** Let $\pi : M \to B$ a Riemannian submersion. If there exists a Lie group action of $\mathbb{R}^k$ on M^n such that π equals the projection map $M \to M/\mathbb{R}^k$, π is called *homogeneous.*

Our goal is to show that for $k \leq 3$, the situation for Riemannian submersions $\pi : \mathbb{R}^{n+k} \to M^n$ is rigid, and that the submersion is given by generalized glide rotations, meaning that there exists a Lie group homomorphism $\varphi : \mathbb{R}^k \to \mathrm{SO}(n)$ such that π is the orbit fibration of the action of $\mathbb{R}^k$ on $\mathbb{R}^{n+k}$ given by

$$u.(x,v) = (\varphi(u)x, v+u).$$

1.2 Foliations

Although we are *a priori* only interested in submersions, at some point we need to generalize the arguments to foliations. Informally speaking, a foliation is a way of dividing a manifold into smooth submanifolds that fit together in a particularly nice way, namely like sheets in a block of paper. In this situation submersions naturally arise locally and we can talk about Riemannian foliations. We will now formally define a foliation, give a criterion – *involutivity* – to recognize these and extend the notion of a Riemannian submersion to these.

Definition 1.14 (Foliation)**.** Let M be a n-dimensional smooth manifold. A *k-dimensional foliation $\mathcal{F}$ of M* is a collection of smooth k-dimensional submanifolds, called *leaves*, such that for every $p \in M$ there exists a chart (U, x) satisfying the following:

(i) $x(U) = V^k \times V^{n-k}$, where $V^k \subset \mathbb{R}^k$, $V^{n-k} \subset \mathbb{R}^{n-k}$ are connected open sets

(ii) every leaf $L \in \mathcal{F}$ intersects U nowhere, or a countable union of k-dimensional slices, i.e. subsets W such that $x(W) = V^k \times \{c\}$, with V^k as above and $c \in \mathbb{R}^{n-k}$.

Clearly if one has a foliation on M, the tangent spaces of the leaves define a distribution. Conversely, given a distribution D, one can ask the question, whether it *generates a foliation*, i.e. if there exists a foliation $\mathcal{F}$ of M such that the distribution defined by the leaves coincides with D. More formally this means, that for any $p \in M$ we have $D_p = T_pL$, where L is the leaf through p. In this situation wie call the foliation *integrable*.

This question is answered by the following standard theorem of smooth manifold theory. For a proof we refer the reader to standard literature, e.g. [Lee13].

Theorem 1.15 (Frobenius' Theorem)**.** Let $D \subset TM$ be a distribution of dimension k. We say the distribution is *involutive*, if for any sections X, Y of D, the Lie bracket $[X, Y]$ is also a section of D. A distribution is integrable iff it is involutive.

Remark 1.16. A foliation $\mathcal{F}$ of M is locally given by submersions.

Proof. Let $p \in M$, (U, x) be a chart around p. With the projection on the first factor, $\mathrm{pr} : \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$, the map $\pi := \mathrm{pr} \circ x$ is clearly a smooth submersion. □

Definition 1.17 (Riemannian foliations)**.** A foliation $\mathcal{F}$ of M is said to be *Riemannian* or *metric* if the local submersions defined by $\mathcal{F}$ are Riemmanian submersions. According to Lemma 1.9 and its corollary this is equivalent to saying that $\mathcal{L}_U g^h \equiv 0$ for all vertical U or equivalently, that ∇^v is skew-symmetric.

Definition 1.18 (Substantial foliations)**.** $\mathcal{F}$ is called *substantial along a leaf* $L \in \mathcal{F}$ if A_X maps the horizontal distribution onto the vertical distribution for some horizontal vector field X. This is equivalent to saying that A_X^* is injective. We say $\mathcal{F}$ is *weakly substantial* if any vertical U is in the image of A_X for some horizontal X.

Definition 1.19 (Flat foliations)**.** The foliation is said to be *flat* if the A-tensor vanishes identically. This means that the horizontal distribution is involutive, and hence by Frobenius' Theorem integrable.

1.3 Holonomy diffeomorphisms and projectable Jacobi fields

We now want to investigate the relationship of Jacobi fields along horizontal geodesics in M and their corresponding projections in B. In particular, we want to give a criterion whether the projection is again a Jacobi field. O'Neill studied this question already in [O'N67], we, however, will follow

Gromoll and Walschap in [GW09]. This approach more geometrical, and relies on the holonomy between fibers along horizontal geodesics.

Definition 1.20 (Holonomy diffeomorphisms)**.** Let $\pi : M \to B$ be a Riemannian submersion, $c : [0,1] \to B$ a geodesic. The *holonomy diffeomorphism* $h : \pi^{-1}(c(0)) \to \pi^{-1}(c(1))$ induced by c is given by

$$h(p) = c_p(1),$$

where c_p is a horizontal lift of c emanating from p.

Remark 1.21. On a foliation the holonomy diffeomorphisms are only locally diffeomorphisms.

Note that the differential of holonomy diffeomorphisms has a quite useful description in terms of Jacobi fields. Let h be as in the lemma, u be a vertical vector at p and X be a basic lift of $\dot{c}(0)$ along $L := \pi^{-1}(c(0))$. Let $\gamma : (-\varepsilon, \varepsilon) \to L$ be a short curve with $\dot{\gamma}(0) = u$. Then we can consider the geodesic variation

$$c_p^s(t) := \exp_{\gamma(s)} tX,$$

and observe that $h \circ \gamma(s) = c_p^s(1)$. In particular, $h_{*p}u = \frac{\partial}{\partial s}\big|_{s=0} c_p^s(1)$. But $c_p^s(t)$ is a geodesic variation, so we have

$$h_{*p}u = J(1),$$

where J is the Jacobi field along $c_p(t)$ with initial conditions $J(0) = u$ and

$$\begin{aligned} J'(0) &= \frac{\nabla}{\partial t}\bigg|_{t=0} \frac{\partial}{\partial s}\bigg|_{s=0} c_p^s(t) = \frac{\nabla}{\partial s}\bigg|_{s=0} \frac{\partial}{\partial t}\bigg|_{t=0} c_p^s(t) \\ &= \frac{\nabla}{\partial s}\bigg|_{s=0} X(\gamma(s)) = \nabla_u X. \end{aligned}$$

Finally observe that

$$J'(0) = \nabla_u X = \nabla_u^v X + \nabla_u^h X = -S_{\dot{c}_p(0)} u - A^*_{\dot{c}_p(0)} u.$$

This motivates the following.

Definition 1.22 (Holonomy fields)**.** A Jacobi field J along a horizontal geodesic $c : [0,a] \to M$, which is vertical at 0 and satisfies

$$J'(0) = -A^*_{\dot{c}(0)} J(0) - S_{\dot{c}(0)} J(0)$$

is called a *holonomy field.* Note that the restriction to $[a', a]$ is also a holonomy field, since it describes the differential of the holonomy diffeomorphism between the fibers through $c(a')$ and $c(a)$. Therefore, the equation holds for all $t \in [0,a]$. Also, because the holonomy displacement is a local diffeomorphism, a holonomy field that vanishes at one point is zero everywhere. Furthermore, such a field is alway vertical.

Definition 1.23 (Projectable fields)**.** Let $c : [0, a] \to M$ be a horizontal geodesic. A Jacobi field J along c is said to be *projectable* if it satisfies

$$J'^v = -S_{\dot{c}} J^v - A_{\dot{c}} J^h.$$

Observe that the collection of projectable Jacobi fields along c is a vector space, containing the collection of holonomy fields as a subspace, since these are vertical and the vertical part of their derivative is exactly $-S_{\dot{c}} J$.

Proposition 1.24 (Variational fields of horizontal geodesics)**.** The variational field of a variation of a geodesic $c : [0, a] \to M$ through horizontal geodesics is projectable.

Proof. Let $c_s(t) : [0, a] \times (-\varepsilon, \varepsilon) \to M$ be a variation of c through horizontal geodesics with $c_0 = c$. Fix some $t_0 \in [0, a]$, let $\gamma(s) = c_s(t_0)$ and denote by $\dot{\gamma}(s)$ its derivative w.r.t. s. Then the variational field of c_s at t_0 is the Jacobi field $J(t_0) = \frac{\partial}{\partial s}\big|_{s=0} c_s(t_0) = \dot{\gamma}(0)$ and we have

$$\begin{aligned} J'(t_0) &= \frac{\nabla}{\partial t}\bigg|_{t=t_0} \frac{\partial}{\partial s}\bigg|_{s=0} c_s(t) = \frac{\nabla}{\partial s}\bigg|_{s=0} \frac{\partial}{\partial t}\bigg|_{t=t_0} c_s(t) \\ &= \frac{\nabla}{\partial s}\bigg|_{s=0} \dot{c}_s(t) = \nabla_{\dot{\gamma}} \dot{c}(t_0) \end{aligned}$$

The variation is through horizontal geodesics, so $\dot{c}_s(t)$ is horizontal. In particular

$$\begin{aligned} J'(t_0)^v &= \nabla^v_{\dot{\gamma}} \dot{c}(t_0)^h = \nabla^v_{\dot{\gamma}^h} \dot{c}(t_0)^h + \nabla^v_{\dot{\gamma}^v} \dot{c}(t_0)^v = \nabla^v_{J^h} \dot{c}(t_0) + \nabla^v_{J^v} \dot{c}(t_0) \\ &= A_{J^h} \dot{c}(t_0) - S_{\dot{c}(t_0)} J^v = -S_{\dot{c}(t_0)} J^v - A_{\dot{c}(t_0)} J^h. \end{aligned}$$

□

Lemma 1.25 (Projectable lifts)**.** Let $c : [0, a] \to M$ a horizontal geodesic and J a Jacobi field along $\pi \circ c$. Then, for any vertical $u \in T_{c(0)}M$, there exists a unique projectable Jacobi field $\tilde{J}$ along c such that

(i) $\pi_* \tilde{J} = J$, and

(ii) $\tilde{J}(0)^v = u$.

Proof. Let $\gamma : (-\varepsilon, \varepsilon) \to B$ be a curve such that $\gamma(0) = \pi \circ c(0)$, $\dot{\gamma}(0) = J(0)$ and V, W parallel along γ such that $V(0) = (\pi \circ c)'(0)$, $W(0) = J'(0)$. Consider the geodesic variation $c_s : [0, a] \times I \to B$, $c_s(t) = \exp_{\gamma(s)} t(V + sW)(s)$ of $\pi \circ c$. Note that $\frac{\partial}{\partial}\big|_{s=0} c_s(0) = J(0)$, and

$$\frac{\nabla}{\partial t}\bigg|_{t=0} \frac{\partial}{\partial s}\bigg|_{s=0} c_s(t) = \frac{\nabla}{\partial s}\bigg|_{s=0} \frac{\partial}{\partial t}\bigg|_{t=0} c_s(t) = \frac{\nabla}{\partial s}\bigg|_{s=0} (V + sW)(s) = J'(0).$$

Let $\tilde\gamma$ be a lift of γ such that $\dot{\tilde\gamma}(0)^v = u$ and $\pi_*\tilde\gamma = \gamma$. Consider the basic lift X_s of $(V+sW)(s)$ along $\tilde\gamma$, and let $\tilde c_s(t) = \exp_{\tilde\gamma(s)} t(X_s \circ \tilde\gamma)(s)$. This is a variation by horizontal geodesics, hence its variational field $\tilde J$ is projectable. Also $\tilde J(0)^v = u$ by construction. Furthermore $\pi \circ \tilde c_s(t) = c_s(t)$, hence

$$\pi_* \tilde J = \pi_* \left.\frac{\partial}{\partial s}\right|_{s=0} \tilde c_s = \left.\frac{\partial}{\partial s}\right|_{s=0} c_s = J.$$

Because $\tilde J$ is projectable and $\tilde J^{h\prime v} = \nabla^v_{\dot c}\tilde J^h = A_{\dot c}\tilde J^h$,

$$\tilde J^{v\prime v} = \tilde J^{\prime v} - \tilde J^{h\prime v} = -S_{\dot c}\tilde J^v - 2A_{\dot c}\tilde J^h.$$

It follows that

$$\tilde J^{v\prime} = \tilde J^{v\prime v} + \tilde J^{v\prime h} = -S_{\dot c}\tilde J^v - 2A_{\dot c}\tilde J^h - A^*_{\dot c}\tilde J^v.$$

This together with (ii) determines $\tilde J^v$ uniquely and by (i) $\tilde J^h$ is determined, so $\tilde J'(0)$ is determined. Also, $\tilde J(0) = u$, so the uniqueness of $\tilde J$ follows, as Jacobi fields are determined by their initial conditions. □

Now we are in a position to give criterion, as to when a Jacobi field along a horizontal geodesic in M projects down to a Jacobi field along the corresponding geodesic in B.

Theorem 1.26 (Projectable fields project to Jacobi fields)**.** *Let $c : [0,a] \to M$ be a horizontal geodesic and be J a projectable Jacobi field along c. Then $\pi_* J$ is a Jacobi field along $\pi \circ c$.*

Proof. Let $\mathcal{P}$ be the space of projectable Jacobi fields along c and $\mathcal{I}$ the space of Jacobi fields along $\pi \circ c$. Consider the map

$$\begin{aligned} \mathcal{V}_{c(0)} \times \mathcal{I} &\to \mathcal{P} \\ (u, J) &\mapsto \tilde J \end{aligned}$$

which maps a vertical vector u and a Jacobi field J along $\pi \circ c$ to the unique projectable Jacobi field $\tilde J$ with $\pi_*\tilde J = J$ and $\tilde J^v(0) = u$. It is well defined by Lemma 1.25. Clearly this map is linear and has by construction a trivial kernel. So $\dim \mathcal{P} \geq k + \dim \mathcal{I}$, where k is the dimension of the fiber. Since $\mathcal{I}$ is the space of Jacobi fields along $\pi \circ c$, its dimension is $\dim \mathcal{I} = 2\dim B$. Hence $\dim \mathcal{P} = k + 2\dim B = \dim M + \dim B$, where k is the dimension of the fiber.

Conversely, consider the map

$$\begin{aligned} \mathcal{P} &\to T_{c(0)}M \times \mathcal{H}_{c(0)} \\ J &\mapsto (J(0), J'^h(0)). \end{aligned}$$

Again, this map is linear. If J is in the kernel of the map, then $J'^h(0) = 0$. Because J is projectable, we have $J'^v(0) = -S_{\dot{c(0)}} J^v(0) - A^*_{\dot{c}(0)} J^h(0)$. But the latter is zero, because $J(0) = 0$ and hence $J(0) = J'(0) = 0$ and the kernel is trivial. In particular, $\dim \mathcal{P} \leq \dim M + \dim B$ and the first map given is an isomorphism, hence the theorem follows. $\square$

The theory of holonomy fields already allows us to establish the following general result.

Lemma 1.27 (A Riccati type equation)**.** Along any horizontal geodesic c, we have the following Riccati type equation:

$$S'^v_{\dot{c}} = S^2_{\dot{c}} - A_{\dot{c}} A^*_{\dot{c}} + R^v_{\dot{c}},$$

where $R^v_{\dot{c}} = R^v(\cdot, \dot{c})\dot{c}$.

Proof. Note that for any t_0 there exists an orthonormal basis of holonomy fields that span the vertical space at $c(t_0)$. So let T be any vertical field along c and J be a holonomy field. Using the symmetries of the Riemannian curvature tensor and the identity for Jacobi fields we have $\langle R(T, \dot{c})\dot{c}, J\rangle = -\langle J'', T\rangle$. Since J is a holonomy field we have

$$\begin{aligned}
-\langle T, J''\rangle &= \langle T, (A^*_{\dot{c}} J)'^v\rangle + \langle T, (S_{\dot{c}} J)'^v\rangle \\
&= \langle T, A_{\dot{c}} A^*_{\dot{c}} J\rangle + \langle T, S_{\dot{c}} J\rangle' - \langle T'^v, S_{\dot{c}} J\rangle \\
&= \langle A_{\dot{c}} A^*_{\dot{c}} T, J\rangle + \langle S_{\dot{c}} T, J\rangle' - \langle S_{\dot{c}} T'^v, J\rangle \\
&= \langle A_{\dot{c}} A^*_{\dot{c}} T, J\rangle + \langle (S_{\dot{c}} T)', J\rangle + \langle S_{\dot{c}} T, S_{\dot{c}} J\rangle - \langle S_{\dot{c}} T'^v, J\rangle \\
&= \langle A_{\dot{c}} A^*_{\dot{c}} T, J\rangle + \langle (S_{\dot{c}} T)', J\rangle + \langle S^2_{\dot{c}} T, J\rangle - \langle S_{\dot{c}} T'^v, J\rangle. \\
&= \langle A_{\dot{c}} A^*_{\dot{c}} T, J\rangle + \langle S^2_{\dot{c}} T, J\rangle + \langle (S'_{\dot{c}} T)^v, J\rangle
\end{aligned}$$

Now it is a simple matter of rearranging the terms to obtain the desired Riccati type equation. Since for any t_0 we can find holonomy fields that form an orthonormal base of the vertical space at $\dot{c}(t_0)$, the Lemma follows. $\square$

1.4 The mean curvature form

Definition 1.28 (Mean curvature form)**.** The *mean curvature form* κ of a Riemannian submersion $\pi : M \to B$ is given by $\kappa(E) = \operatorname{tr} S_{E^h}$, with $E \in \mathcal{V}M$.

Lemma 1.29. Let T_i be a local orthonormal frame of the vertical distribution. Then we have the following description of the mean curvature form:

$$\kappa(E) = \operatorname{tr} S_{E^h} = \sum_i \langle -\nabla^v_{T_i} E^h, T_i\rangle = \sum_i \langle \nabla_{T_i} T_i, E^h\rangle = \sum_i \langle T_i, [E^h, T_i]\rangle$$

Proof. The first equality is just the definition. For the second equality, observe that $\langle -\nabla_{T_i} E^h, T_i \rangle = -T_i \langle E^h, T_i \rangle + \langle \nabla_{T_i} T_i, E^h \rangle$. But since E^h and T_i are orthogonal to each other, the first term vanishes.

Similarly, we have for any X, $0 = X\langle T_i, T_i \rangle = 2\langle T_i, \nabla_X T_i \rangle$, hence $\langle -\nabla^v_{T_i} E^h, T_i \rangle = \langle -\nabla_{E^h} T_i - [T_i, E^h], T_i \rangle = \langle [E^h, T_i], T_i \rangle$. □

Lemma 1.30. For basic X, Y the exterior derivative $d\kappa(X, Y)$ is given by $-2 \operatorname{div} A_X Y$.

Proof. Recall that $d\kappa(X, Y) = X(\kappa Y) - Y(\kappa X) - \kappa[X, Y]$, so the claim will follow once we establish that

$$\kappa([X, Y]) = \kappa([X, Y]^h) = X(\kappa Y) - Y(\kappa X) + 2 \operatorname{div} A_X Y.$$

First want to see the origin of the divergence term:

$$\begin{aligned}
\kappa[X, Y] = \kappa([X, Y]^h) &= \sum_i \langle [[X, Y]^h, T_i], T_i \rangle \\
&= \sum_i \{\langle [[X, Y], T_i] - [[X, Y]^v, T_i], T_i \rangle\} \\
&= \sum_i \{\langle [[X, Y], T_i], T_i \rangle + \langle T_i, \nabla_{T_i} [X, Y]^v \rangle\}
\end{aligned}$$

The last equality follows from the fact that the T_i form an orthonormal base, so $0 = X\langle T_i, T_i \rangle = 2\langle T_i, \nabla_X T_i \rangle$. Note that for X basic, T vertical $\langle \nabla_X T, X \rangle = \langle \nabla_T X, X \rangle + \langle [X, T], X \rangle$. As observed in the proof of the preceding lemma, the first term vanishes, and the second term is zero, since $[X, T]$ is vertical for basic X. It follows immediately that the divergence of a vertical field is just its divergence in the leaf, so

$$\begin{aligned}
\kappa[X, Y] &= \operatorname{div}([X, Y]^v) + \sum_i \langle [[X, Y], T_i], T_i \rangle \\
&= 2 \operatorname{div} A_X Y + \sum_i \langle [[X, Y], T_i], T_i \rangle.
\end{aligned}$$

It remains to show that $\sum_i \langle [[X, Y], T_i], T_i \rangle = X(\kappa Y) - Y(\kappa X)$. To this end let $X_{ij} = \langle [X, T_i], T_j \rangle$, $Y_{ij} = \langle [Y, T_i], T_j \rangle$, so we have $[X, T_i] = \sum_j X_{ij} T_j$ and $[Y, T_i] = \sum_j Y_{ij} T_j$.

Using the Jacobi identity, this convention and the skew-symmetry of the bracket, we obtain

$$\begin{aligned}\sum_i \langle [[X,Y],T_i],T_i\rangle &= \sum_i \{\langle [[X,T_i],Y],T_i\rangle - \langle [[Y,T_i],X],T_i\rangle\} \\ &= \sum_i \left\langle \left[\sum_j X_{ij}T_j, Y\right], T_i\right\rangle \\ &- \sum_i \left\langle \left[\sum_j Y_{ij}T_j, X\right], T_i\right\rangle \\ &= \sum_{i,j} \{\langle [X_{ij}T_j, Y], T_i\rangle - \langle [Y_{ij}T_j, X], T_i\rangle\}.\end{aligned}$$

Since the X_{ij} and Y_{ij} are smooth functions, the Leibniz rule for the Lie bracket applies:

$$\begin{aligned}\sum_i \langle [[X,Y],T_i],T_i\rangle &= \sum_{i,j} \{X_{ij}\langle [T_j,Y],T_i\rangle - Y(X_{ij}\langle T_j,T_i\rangle \\ &- Y_{ij}\langle [T_j,X],T_i\rangle + X(Y_{ij})\langle T_j,T_i\rangle\} \\ &= \sum_{i,j} X_{ij}Y_{ji} - Y_{ij}X_{ji} + \sum_i X(Y_{ii}) - Y(X_{ii})\end{aligned}$$

We sum up over all pairs (i,j), so the first sum is zero. Therefore,

$$\begin{aligned}\sum_i \langle [[X,Y],T_i],T_i\rangle &= \sum_i \{X(Y_{ii}) - Y(X_{ii})\} \\ &= \sum_i \{X\langle [Y,T_i],T_i\rangle - Y\langle [X,T_i],T_i\rangle\} \\ &= X\kappa(Y) - Y\kappa(X),\end{aligned}$$

and the Lemma follows from the observation made in the beginning. □

Definition 1.31 (Isoparametricity)**.** A metric foliation is said to be *isoparametric*, if the principal curvatures in basic directions are locally constant along leaves.

Theorem 1.32 (Foliations of $\mathbb{R}^n$ are isoparametric)**.** All metric foliations of $\mathbb{R}^n$ are isoparametric. In particular, their mean curvature form is basic.

Proof. Let $p \in \mathbb{R}^n$, $x \in T_p\mathbb{R}^n$ be horizontal and U be an open neighbourhood of p such that $\pi : U \to B$ is a Riemannian submersion defining the foliation locally.

We prove a slightly stronger statement: If λ is an eigenvalue of S_x, then λ is also an eigenvalue of the same multiplicity of $S_{\tilde{x}}$, where $\tilde{x}$ is any horizontal vector with $\pi_* x = \pi_* \tilde{x}$. Let γ and $\tilde{\gamma}$ be horizontal geodesics with initial vectors x and $\tilde{x}$ respectively. Let u be a unit λ-eigenvector of S_x, i.e. $S_x u = \lambda u$. Consider the Jacobi field J along γ with initial conditions $J(0) = u$, $J'(0) = -S_{\dot{\gamma}(0)} u = -\lambda u$. Because u is vertical, this is a projectable Jacobi field, hence $\pi_* J$ is Jacobi along $\pi \circ \gamma$. Furthermore, because of the form of Jacobi fields in $\mathbb{R}^n$, $J(t) = (1 - \lambda t) E(t)$, where $E(t)$ is a parallel extension of u along γ.

Assume for now that $\lambda \neq 0$. Then $J(l) = 0$ for $l = \frac{1}{\lambda}$. In particular $\pi_* J(0) = \pi_* J(l) = 0$. By Lemma 1.25 there exists a unique projectable Jacobi field $\tilde{J}$ along $\tilde{\gamma}$ such that $\pi_* \tilde{J} = \pi_* J$. Now $\pi_* J(0) = 0$, so $\tilde{J}(0)$ must also be vertical. Hence $\tilde{J}'^v(0) = -S_{\tilde{x}} \tilde{J}(0)$. This implies that $\tilde{J}(t) = (1 - \lambda t) \tilde{E}(t)$, where $\tilde{E}$ is a parallel extension of $\tilde{J}(0)$ along $\tilde{\gamma}$. In particular $\tilde{J}'(0) = -\lambda \tilde{E}(0)$, so $\tilde{E}(0)$ is a λ-eigenvalue of $S_{\tilde{x}}$. But by [O'N67, Theorem 4] the order of $\gamma(l)$ and $\tilde{\gamma}(l)$ as focal points of $\gamma(0)$, $\bar{\gamma}(0)$ along γ and $\tilde{\gamma}$ coincide (being equal to the order of conjugacy downstairs), so the λ-eigenspaces of $S_{\tilde{x}}$ and S_x have the same dimension. So for non zero eigenvalues, the statement is true. But then it must also be true for $\lambda = 0$. □

Definition 1.33 (Minimal foliations)**.** A foliation is called *by minimal leaves* or simply *minimal* if $\kappa \equiv 0$.

Lemma 1.34 (Minimal foliations of $\mathbb{R}^n$)**.** Any codimension k Riemannian foliation of $\mathbb{R}^n$ by minimal leaves is totally geodesic and flat.

Proof. Consider the Riccati type equation of Lemma 1.27. In Euclidean space the equation reduces to $S_{\dot{c}}'^v = S_{\dot{c}}^2 - A_{\dot{c}} A_{\dot{c}}^*$. Taking traces we obtain

$$(\operatorname{tr} S_{\dot{c}})' = \|S_{\dot{c}}\|^2 - \|A_{\dot{c}}\|^2.$$

To see this, let T_i be a parallel orthonormal frame of the vertical distribution along c. Then

$$\operatorname{tr} S_{\dot{c}}'^v = \sum_i \langle (S_{\dot{c}} T_i)' - (S_{\dot{c}}(T_i'^v), T_i \rangle,$$

hence the last term vanishes, because the T_i are chosen parallel along c. Similarily,

$$(\operatorname{tr} S_{\dot{c}})' = \sum_i \langle S_{\dot{c}} T_i, T_i \rangle' = \sum_i \langle (S_{\dot{c}} T_i)', T_i \rangle.$$

The right hand side follows immediately from the fact that $S_{\dot{c}}$ is self-adjoint and the general fact that for adjoint operators A and A^* one has $\|A^*\|^2 = \|A\|^2$:

$$\operatorname{tr}(S_{\dot{c}}^2 - A_{\dot{c}} A_{\dot{c}}^*) = \sum_i \langle S_{\dot{c}} T_i, S_{\dot{c}} T_i \rangle - \sum_i \langle A_{\dot{c}}^* T_i, A_{\dot{c}}^* T_i \rangle = \|S_{\dot{c}}\|^2 - \|A_{\dot{c}}\|^2$$

It follows that $(\operatorname{tr} S_{\dot{c}})' = \|S_{\dot{c}}\|^2 - \|A_{\dot{c}}\|^2$, and by assumption $(\operatorname{tr} S_{\dot{c}})' = \kappa(\dot{c})' = 0$. Hence $\|S_{\dot{c}}\|^2 = \|A_{\dot{c}}^*\|^2$.

Observe that $\ker(S_{\dot{c}} + A^*_{\dot{c}})$ is a parallel subspace along c: If u belongs to the kernel at, say $t = 0$, then the holonomy field J with $J(0) = u$ is parallel, because $J'(0) = 0$.

Towards a contradiction, assume that $\mathrm{im}(S_{\dot{c}} + A^*_{\dot{c}}) \neq \{0\}$. Let f_i be an orthonormal basis of $\mathrm{im}(S_{\dot{c}(0)} + A^*_{\dot{c}(0)})$ and e_i vectors such that $(S_{\dot{c}(0)} + A^*_{\dot{c}(0)})e_i = -f_i$. Let E_i and F_i be their parallel extensions along c. Then the holonomy fields J_i with $J_i(0) = e_i$ are given by $J_i(t) = E_i(t) + tF_i(t)$.

Let J be such a holonomy field, and let $U = \frac{J}{\|J\|}$. Then $S_{\dot{c}}U = \frac{1}{\|J\|}S_{\dot{c}}J = \frac{J'^v}{\|J\|}$. U is a unit vertical vector field. Let V_i be unit vector fields such that $U, V_1, \ldots, V_{k-1}$ form an orthonormal base of the vertical distribution. Then

$$J'^v = \langle J', U\rangle U + \sum_{i=1}^{k-1} \langle J', V_i\rangle V_i,$$

and, because the summands are orthogonal to each other,

$$\|J'^v\|^2 = \|\langle J', U\rangle U\|^2 + \sum_{i=1}^{k-1} \|\langle J', V_i\rangle V_i\|^2 = \left(\frac{\langle J', J\rangle}{\|J\|^2}\right)^2 + \sum_{i=1}^{k} \|\langle J', V_i\rangle V_i\|^2.$$

It follows that

$$\|S_{\dot{c}}U\|^2 = \frac{1}{\|J\|^2}\|S_{\dot{c}}J\|^2 \geq \left(\frac{\langle J', J\rangle}{\|J\|^2}\right)^2 = \frac{\left(\langle E, F\rangle + t\|F\|^2\right)^2}{(\|E\|^2 + 2t\langle E, F\rangle + t^2\|F\|^2)^2},$$

hence $t^2\|S_{\dot{c}}U\|^2 \to a$ for some $a \geq 1$ as t tends to infinity. Similarly,

$$t^2\frac{\|J'\|^2}{\|J\|^2} = t^2\frac{\|F\|^2}{\|E\|^2 + 2t\langle E, F\rangle + t^2\|F\|^2} \to 1.$$

Since J is a holonomy field, we have

$$\|A^*_{\dot{c}}U\|^2 + \|S_{\dot{c}}U\|^2 = \frac{1}{\|J\|^2}\left(\|A^*_{\dot{c}}J\|^2 + \|S_{\dot{c}}J\|^2\right) = \frac{\|J'\|^2}{\|J\|^2}.$$

Consequently we have $t^2\|A_{\dot{c}}U\|^2 \to 0$ for $t \to \infty$. In particular we have

$$\frac{\|A_{\dot{c}\|^2}}{\|S_{\dot{c}}\|^2} \to 0,$$

but this is a contradiction to $\|A_{\dot{c}}\|^2 \equiv \|S_{\dot{c}}\|^2$. Hence $(S_{\dot{c}j} + A^*_{\dot{c}})$ must have trivial image and any vertical vector u must belong to the kernel. Since $S_{\dot{c}}u \perp A^*_{\dot{c}}u$ the claim follows. □

Remark 1.35 (A more concise proof)**.** In the case of a fibration, the proof of Lemma 1.34 becomes a lot simpler. As we will see, in this case there is a totally geodesic fiber F, so along F one has $S_{\dot{c}} \equiv 0$. Then the Riccati type equation yields that along F we also have $A_{\dot{c}} \equiv 0$. But if the A-tensor vanishes at some point p in some connected space of constant curvature K, it vanishes everywhere: By connectedness it suffices to show that $A \equiv 0$ in a neighbourhood of p. By O'Neill's formula (Theorem 1.11), $\|A_X Y\|$ is constant along a leaf L, so we have to show that $A_X Y$ is zero along any horizontal geodesic γ emanating from p. Indeed, any parallel vector field along γ with $E(0)$ horizontal stays horizontal. So the claim follows once we have established this, since $A_X Y_p$ is horizontal. If J is a holonomy field along γ and E parallel, we have

$$\langle J, E\rangle'' = \langle J, E'\rangle' + \langle J', E\rangle' = \langle J'', E\rangle = -\langle R(J, \dot{\gamma})\dot{\gamma}, E\rangle = -K\langle J, E\rangle.$$

Since J is a holonomy field

$$\langle J, E\rangle' = \langle J', E\rangle = -\langle (S_{\dot{\gamma}} + A^*_{\dot{\gamma}})J, E\rangle = -\langle A^*_{\dot{\gamma}} J, E\rangle.$$

Now $\langle J, E\rangle_p = 0$ by assumption and $\langle A^*_{\dot{\gamma}} J, E\rangle_p = \langle J, A_{\dot{\gamma}} E\rangle_p = 0$, hence the claim follows.

Also note that by [FGLT00, Theorem 1.7], any foliation of Euclidean space is given by a submersion, so we actually always are in the above situation.

1.5 The Bott connection

We now introduce the *Bott connection*, a notion that plays a major role in understanding the foliation.

Definition 1.36 (The Bott connection)**.** Let $\mathcal{F}$ be a foliation of a Riemannian manifold M^n, and L a leaf. The *Bott connection* is given by

$$\begin{aligned} \nabla^B : TL \times \Gamma\nu L &\to \Gamma\nu L \\ (u, X) &\mapsto [U, X]^h, \end{aligned}$$

where U is any vertical extension of u; this definition makes sense, as $[U, X]^h$ is tensorial in the first argument, since for any smooth φ one has

$$[\varphi U, X]^h = \varphi[U, X]^h - X(\varphi)U^h = \varphi[U, X]^h.$$

It follows immediately from the properties of the Lie bracket that this is indeed a connection.

We now proceed to show two important lemmata which relate the Bott connection to the foliation. The first one tells us that the foliation is metric iff the Bott connection is. The second one characterizes the basic vector fields as those which are Bott parallel.

Lemma 1.37 (A characterization of metric foliations)**.** Let $\mathcal{F}$ be a foliation of a Riemannian manifold M^n. The foliation is metric if and only if the Bott connection is metric.

Proof. This is an immediate result of Lemma 1.9: We have

$$Ug(X,Y) = \mathcal{L}_U g^h(X,Y) + g([U,X]^h, Y) + g(X, [U,Y]^h).$$

But $\mathcal{F}$ is metric iff $\mathcal{L}_U g^h(X,Y)$ vanishes. □

Remark 1.38 (Flatness of the Bott connection)**.** The Bott connection is flat.

Proof. Let U, V be vertical, X be horizontal. Then the curvature of ∇^B is given by

$$\begin{aligned} R^B(U,V)X &= [U,[V,X]^h]^h - [V,[U,X]^h]^h - [[U,V],X]^h \\ &= [U,[V,X]^h]^h + [X,[U,V]]^h + [V,[X,U]^h]^h. \end{aligned}$$

But in a foliation the vertical distribution is integrable, i.e. $[U,V]$ is vertical. In particular, we have for any vector field E

$$[U,E^h]^h = [U,E-E^v]^h = [U,E]^h - [U,E^v]^h = [U,E]^h,$$

and the claim follows from the Jacobi identity for the Lie bracket. □

Another way to see that the Bott connection is flat would have been to note that it admits parallel sections, namely the basic vector fields.

Lemma 1.39 (Basic vector fields are Bott parallel)**.** In case of a metric foliation, the basic vector fields on M are exactly the Bott parallel ones.

Proof. We have already seen that for vertical U and basic X, $[U,X]$ is vertical, so basic vector fields are parallel w.r.t. the Bott connection. On the other hand, let U be a vertical and X be a horizontal field such that $[U,X]^h = 0$. Let $X_1, \dots, X_n$ be a base of basic vector fields as above, write $X = \sum_{i=1}^n \varphi_i X_i$. Then

$$\begin{aligned} 0 = [U,X]^h = [U, \sum_{i=1}^n \varphi_i X_i]^h &= \left(\sum_{i=1}^n [U, \varphi_i X_i] \right)^h \\ &= \left(\sum_{i=1}^n U(\varphi_i) X_i + \varphi_i \underbrace{[U, X_i]}_{\text{vertical}} \right)^h = \sum_{i=1}^n U(\varphi_i) X_i. \end{aligned}$$

Therefore $U(\varphi_i) = 0$ for all i, and hence X is basic. □

To conclude, we introduce the connection difference form; it will play a crucial role in establishing the main theorem. As it turns out, it will describe the differential of the Lie group homomorphism.

Definition 1.40 (The connection difference form)**.** The *connection difference form* ω is the 1-form given by

$$\omega(U)X = (\nabla_U X)^h - \nabla_U^B X.$$

Observe that $\omega(U)X = -A_X^* U$, and in particular, that ω takes values in the skew-symmetric endomorphism bundle. To see this, let Y be any horizontal vector field. Then, as $[U, X]$ is vertical,

$$\begin{aligned}
\langle \omega(U)X, Y \rangle &= \langle (\nabla_U X - [U, X])^h, Y \rangle = \langle (\nabla_X U)^h, Y \rangle \\
&= X\langle U, Y \rangle - \langle U, \nabla_X Y \rangle = -\langle U, \nabla_X Y \rangle \\
&= \langle U, -(\nabla_X Y)^v \rangle = \langle -A_X^* U, Y \rangle.
\end{aligned}$$

2 Riemannian submersions of Euclidean space

We now restrict our attention to Riemannian submersions $\pi : \mathbb{R}^{n+k} \to M^n$, and unless otherwise stated, π will denote such a submersion. Clearly π defines a fibration. From O'Neill's formula (Theorem 1.11), we know that M^n must have nonnegative curvature. Using a little topology, we can obtain some further properties:

Let $F = \pi^{-1}(b)$ the fiber over some point $b \in M$. Then the long sequence of homotopy groups

$$\dots \longrightarrow \pi_n(F) \xrightarrow{i_*} \pi_n(\mathbb{R}^{n+k}) \xrightarrow{\pi_*} \pi_n(M) \xrightarrow{\partial} \pi_{n-1}(F) \longrightarrow \dots$$

is exact, where $i : F \to \mathbb{R}^{n+k}$ denotes the inclusion of F in the total space and ∂ is the boundary operator. This yields the following.

Observation 2.1 (Connectedness of the fibers)**.** The fibers F are connected if and only if M is simply connected.

Proof. For the arrow between $\pi_1(M)$ and $\pi_0(F)$ we have

$$\pi_1(\mathbb{R}^{n+k}) \xrightarrow{\pi_*} \pi_1(M) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(\mathbb{R}^{n+k}) \xrightarrow{\pi_*} \pi_0(M) .$$

But $\pi_1(\mathbb{R}^{n+k}) \cong 0$ and $\pi_0(\mathbb{R}^{n+k}) \cong \mathbb{Z}$, so ∂ is injective. Since π is smooth and $\mathbb{R}^{n+k}$ is connected, M is too, and $\pi_0(M) \cong \mathbb{Z}$.

Now assume F is connected, i.e. $\pi_0(F) \cong \mathbb{Z}$. Then clearly i_* is injective. Exactness of the sequence yields $\operatorname{im} \partial = \ker i_* \cong 0$. But ∂ is injective, so we have shown that $\pi_1(M)$ must be 0 and M is simply connected.

If we assume that M is simply connected, it follows that $\operatorname{im} \partial \cong 0$ and hence $\ker i_* \cong 0$. In particular $\pi_0(F) \cong \mathbb{Z}$, since F is non-empty and F is connected. □

It follows from the homology spectral sequence, that M is actually diffeomorphic to Euclidean space (cf. [GP97, Theorem 3.1]).

Now we can and will restrict our attention to the case where the fibers are connected and M is simply connected, since π factors as a fibration over the universal cover $\tilde{M}$ of M and a covering map. But such covering maps are well understood and $\tilde{M}$ is isometric to $M_0 \times \mathbb{R}^m$ for some m, where M_0 is a compact manifold of nonnegative curvature (cf. [CG72]).

2.1 Lifting of the soul construction

Our next goal is to establish that for any submersion $\pi : \mathbb{R}^{n+k} \to M^n$ there is a totally geodesic fiber F^k. While this result is interesting in its own right, an immediate consequence is that the mean curvature form is exact. Furthermore, it plays a key role in establishing homogeneity of the foliation in the next section.

Recall the much celebrated Soul Theorem of Jeff Cheeger and Detleff Gromoll ([CG72]):

Theorem 2.2 (Soul Theorem, Cheeger-Gromoll, 1972)**.** Let $(M, \langle,\rangle)$ be an open manifold with $K \geq 0$. Then there exists a compact, totally geodesic, totally convex submanifold $\Sigma \subset M$ without boundary, the *soul of* M, such that M is diffeomorphic to the normal bundle $\nu(\Sigma)$.

As observed, M has nonnegative curvature and is diffeomorphic to Euclidean space. So M has a soul and this soul must be a point. We proceed to show that the fiber over the soul is totally geodesic and hence an affine subspace of $\mathbb{R}^{n+k}$.

Recall the process of constructing the soul. For a ray γ in M we define $B_\gamma = \bigcup_{t>0} B_t(\gamma(t))$, where $B_r(p)$ is the ball of radius r around p. Then its complement C_γ is a totally convex subset (*t.c.s.*) of M. Fixing a point $p \in M$, we can take the intersection of all C_γ, where γ is a ray emanating from p. This yields a compact t.c.s. C. In fact such an C is a submanifold with totally geodesic interior and possibly a nonsmooth boundary ∂C. If $\partial C = \emptyset$, C is the soul of M. Otherwise let C_1 be the subset of points at maximal distance from ∂C. Again this yields a t.c.s., of strictly lower dimension, with possibly nonsmooth and nonempty boundary ∂C_1. Now iterate this process until at a t.c.s. C_k with $\partial C_k = \emptyset$. This is the soul of M, and consists in our case of single point.

Now we can lift this process to $\mathbb{R}^{n+k}$. If $\bar{\gamma}$ is a horizontal lift of γ to $\mathbb{R}^{n+k}$, we obtain a set $B_{\bar{\gamma}} = \bigcup_{t>0} B_t(\bar{\gamma}(t))$. Because π is a submetry, i.e. maps closed balls of radius r to closed balls of radius r, it is clear that $\pi(B_{\bar{\gamma}}) = B_\gamma$. Let $\bar{B}_\gamma$ be the union of all $B_{\bar{\gamma}}$, where $\bar{\gamma}$ ranges over the horizontal lifts of γ, and let $\bar{C}_\gamma = \mathbb{R}^{n+k} \setminus \bar{B}_\gamma$. Now we show that $\bar{C}_\gamma$ projects down to C_γ.

Lemma 2.3. $\bar{C}_\gamma$ is a closed convex set and $\pi(\bar{C}_\gamma) = C_\gamma$.

Proof. Note that the $B_{\bar{\gamma}}$ are open half-spaces, hence $\bar{C}_\gamma$ is the intersection of closed half-spaces and as such a closed convex set.

First we show that $C_\gamma \subset \pi(\bar{C}_\gamma)$: Let $x \in C_\gamma$, i.e. $x \notin B_\gamma$. Then for $\bar{x} \in \pi^{-1}(x)$ we also have for all lifts $\bar{\gamma}$ of γ $\bar{x} \notin B_{\bar{\gamma}}$, because else $\pi(\bar{x}) = x \in B_\gamma$. So $\bar{x} \in \bar{C}_\gamma$.

For the converse, i.e. $\pi(\bar{C}_\gamma) \subset C_\gamma$ we assume towards a contradiction that this does not hold. Then there exists an $\bar{x} \in \bar{C}_\gamma$ for which $x := \pi(\bar{x}) \in B_\gamma$. This means that there exists $t_0 > 0$ such that $d(\gamma(t_0), x) < t_0$. Let c be a minimal geodesic connecting $\gamma(t_0)$ with x. Then $\bar{c}(0) = \bar{\gamma}(t_0)$ for some horizontal lift $\bar{c}$ of c. But this means that $d(\bar{\gamma}(t_0), \bar{x}) = d(\gamma(t_0), x) < t_0$, a contradiction to $\bar{x} \in \bar{C}_\gamma$. □

Theorem 2.4 (The fiber over a soul)**.** If $\pi : \mathbb{R}^{n+k} \to M^n$ is a Riemannian submersion, then the fiber $\pi^{-1}(p)$ over a soul p of M is an affine subspace.

Proof. Let $\bar{C} = \bigcap_\gamma \bar{C}_\gamma$, where γ ranges over all rays in M^n emanating from p. Clearly $\bar{C}$ is a closed and convex subset of Euclidean space. Indeed, $\bar{C} = \pi^{-1}(C)$, where C is the t.c.s. from the soul construction as described above: If $\bar{x} \in \bar{C}$, then for all rays γ emanating from p $\bar{x} \in \bar{C}_\gamma$, so for all γ, $\pi(\bar{x}) \in C_\gamma$, hence $\pi(\bar{x}) \in C$. Conversely, if $\bar{x} \notin \bar{C}$, then there exists a ray γ emanating from p such that $\bar{x} \notin \bar{C}_\gamma$, i.e. for this γ we have $\bar{x} \in \bar{B}_\gamma$. Hence there exists a lift $\bar{\gamma}$ of γ such that $\bar{x} \in B_{\bar{\gamma}}$. But then $\pi(\bar{x}) \in B_\gamma$, thus $\pi(\bar{x}) \notin C_\gamma$ and in particular $\pi(\bar{x}) \notin C$. Thus $\bar{C} = \pi^{-1}(C)$ and consequently $\partial\bar{C} = \pi^{-1}(\partial C)$.

If the boundary of C is empty, i.e. C is the soul $\{p\}$ of M^n, then $\bar{C}$ is totally geodesic and hence an affine subspace of $\mathbb{R}^{n+k}$.

Otherwise consider for $0 \le a \le a_0 = \max\{d(q, \partial C) \mid q \in C\}$ the sets $C^a = \{q \in C \mid d(q, \partial C) \ge a\}$ and $\bar{C}^a = \left\{q \in \bar{C} \mid d(q, \partial\bar{C}) \ge a\right\}$. Both are superlevel sets of the (in the sense of support functions) concave distance function, so C^a is a closed t.c.s. in M and $\bar{C}^a$ is a closed convex set in $\mathbb{R}^{n+k}$.

Also $\bar{C}^a = \pi^{-1}(C^a)$, because $\bar{x} \in \bar{C}^a$ iff $d(\bar{x}, \bar{b}) \ge a$ for all $\bar{b} \in \partial\bar{C}^a$. But $d(\bar{x}, \bar{b}) = d(x, b)$, where $x = \pi(\bar{x})$, $b = \pi(\bar{b})$, so $d(x, b) \ge a$.

In particular, $C(1) = C^{a_0}$ is the lower-dimensional t.c.s. in the soul construction and $\bar{C}(1) = \bar{C}^{a_0}$, so $\bar{C} = \pi^{-1}(C(1))$. Iterating this procedure yields a set $C(l)$ after finitely many steps, the soul $\{p\}$, and by the above arguments $\bar{C}(l)$ is a closed convex subset without boundary of $\mathbb{R}^{n+k}$, i.e. an affine subspace. □

Observe that the soul $\{p\}$ of M is essentially unique. If there is another soul, say, $\{q\}$, then the totally geodesic fibers F_p and F_q over the souls are equidistant. So the minimal connections between F_p and F_q define parallel sections of their respective normal bundles. Exponentiating these sections yields an affine subspace, which projects down to a line through p and q. So in this case we can split off an euclidean factor $\mathbb{R}$, i.e. $\pi : \mathbb{R}^{n+k-1} \times \mathbb{R} \to M^{n-1} \times \mathbb{R}$, $\pi = (\pi', \mathrm{id})$.

The existence of the affine leaf has the following immediate consequence for the fibration.

Corollary 2.5 (Exact mean curvature form)**.** Any metric fibration of Euclidean space has exact mean curvature form.

Proof. Recall that $\kappa(E) = \operatorname{tr} S_{E^h}$, so for vertical U, V ,clearly ,

$$d\kappa(U, V) = U\kappa(V) - V\kappa(U) - \kappa([U, V]) = 0.$$

Note that, by Theorem 1.32, κ is basic. For basic X we have $[X, U]$ is vertical, so $d\kappa(X, U) = X\kappa(U) - U\kappa(X) - \kappa([X, U])$. But the first and last term are zero, because U and $[X, U]$ are vertical, also the second term is zero, because κ is basic.

It remains to show that $d\kappa(X, Y) = 0$ for basic X, Y. By Lemma 1.30 $d\kappa(X, Y) = -2 \operatorname{div} A_X Y$. This is the divergence induced by the metric on the corresponding fiber: Let $e_1, \ldots, e_{n+k}$ be an orthonormal base of the fiber, with $e_1, \ldots, e_n$ horizontal, $e_{n+1}, \ldots, e_{n+k}$ vertical. Then $\operatorname{div} A_X Y = \sum_{i=1}^{n+k} \langle \nabla_{e_i} A_X Y, e_i \rangle$. If $E_1, \ldots E_n$ are basic extensions of $e_1, \ldots, e_n$, then

$$\langle \nabla_{E_i} A_X Y, E_i \rangle = -\langle A_X Y, \nabla_{E_i} E_i \rangle = 0.$$

Note that $\operatorname{div} A_X Y$ is constant along a fiber, for κ and therefore $d\kappa$ are basic.

Let L be any fiber and c a minimal segment of length l from the totally geodesic fiber F to L. Consider the holonomy diffeomorphism $h^c : F \to L$ arising from horizontal lifts of $\pi \circ c$. Its differential $h^c_* u$ is given by $J(l)$, J being the holonomy field with initial conditions $J(0) = u$, $J'(0) = -A^*_{\dot{c}} u - S_{\dot{c}} u$. But F is totally geodesic, so the last term vanishes. With X being the basic field with $\pi_* \dot{c}(0) = \pi_* X$, the form of Jacobi fields in Euclidean space yields that

$$\|h^c_* u\|^2 = \|u\|^2 + l^2 \|A^*_X u\|^2.$$

So $\|h^c_*\|^2$ is bounded from below by 1. But $\|A_X Y\|^2$ is constant along fibers for basic X, Y, so $\|A^*_X u\|^2$ is bounded from above, and so is $\|h^c_*\|^2$.

Let $B_r \subset L$ be the image of the ball of radius r in F around some fixed point under the diffeomorphism h^c (see Figure 2). Then these bounds imply that $\operatorname{vol} B_r \geq a r^k$ for and $\operatorname{vol} \partial B_r \leq b r^{k-1}$ for some constants a and b.

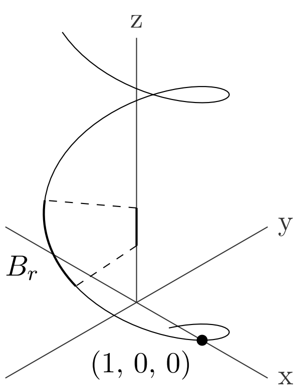

Figure 2: A (here 1-dimensional) ball of small radius r in the totally geodesic fiber gets mapped via holonomy diffeomorphism onto a ball B_r in the fiber L. The differential of the holonomy diffeomorphism yields bounds on its volume.

Using Stokes' Theorem and the fact that $\operatorname{div} A_X Y$ is constant along fibers we have

$$a \|\operatorname{div} A_X Y\| r^k \leq \left| \int_{B_r} \operatorname{div} A_X Y \right| = \left| \int_{\partial B_r} \langle A_X Y, N_r \rangle \right| \leq b \|A_X Y\| r^{k-1},$$

where N_r denotes the unit normal to ∂B_r. But this inequality must hold for all $r > 0$, hence $\operatorname{div} A_X Y$ must be zero on L. $\square$

2.2 Homogeneity in low codimension

In this section, our goal is to prove the following.

Theorem 2.6 (Gromoll-Walschap, 1997)**.** Any k-dimensional metric fibration of $\mathbb{R}^{n+k}$ with $k \leq 3$ is homogeneous. In particular, metric fibrations are up to congruence in 1-1 correspondence with equivalence classes of representations $\mathbb{R}^k \to \mathrm{SO}(n)$.

The proof is similar to the homogeneity result in [GG88] for fibrations of spheres in the way that both approaches rely on basic linear algebra. The proof consists roughly of three steps. First we follow [GW97] and establish the main result Theorem 2.6: If the connection difference $\omega = -A^*$ is basic, or equivalently, that $\langle A_X Y, U \rangle$ is constant for parallel U and basic X, Y iff ω is closed. In particular, ω induces a Lie algebra homomorphism and the corresponding group homomorphism yields the generalized glide rotations. So far we do not need the assumption that $k \leq 3$.

Then we need the assumption $k \leq 3$ to show that $\langle A_X Y, A_Z W \rangle$ is constant along the totally geodesic fiber F^k for X, Y, Z, W basic. Here we follow [GG88] and [GW97].

Finally, going back to [GW97] we use that $\langle A_X Y, A_Z W \rangle$ is constant along F^k to obtain that ω is a Lie algebra homomorphism. Using this and some classification results for Lie algebras, we obtain that ω is exact and can apply Theorem 2.6.

As a first step, note the following general fact which relates the curvature tensors of different connections on a vector bundle.

Lemma 2.7 (Curvature of different connections)**.** Let $E \to M$ be a vector bundle, $\nabla^1, \nabla^2 : \mathcal{V}M \times \Gamma(E) \to \Gamma(E)$ be two connections. Then their curvature tensors R^1 and R^2 are related via

$$R^1(X,Y)\sigma - R^2(X,Y)\sigma = d_{\nabla^2}\omega(X,Y)\sigma + [\omega(X), \omega(Y)]\sigma,$$

where $X, Y \in \mathcal{V}M$ are vector fields on M, $\sigma \in \Gamma(E)$ is a section, $\omega = \nabla^1 - \nabla^2$ is the connection difference 1-form and $d_{\nabla^2}\omega$ is the exterior derivative of ω associated to ∇^2, defined by

$$d_{\nabla^2}\omega(X,Y)\sigma = \nabla^2_X(\omega(Y))\sigma - \nabla^2_Y(\omega(X))\sigma - \omega([X,Y])\sigma.$$

Proof. Recall that a connection ∇ of $E \to M$ extends to 1-forms $\omega : \mathcal{V}M \to \mathrm{End}(\Gamma(E))$ via

$$\nabla_X(\omega(Y))\sigma = \nabla_X(\omega(Y)\sigma) - \omega(Y)\nabla_X\sigma.$$

Using this we calculate:

$$\begin{aligned} d_{\nabla^2}\omega(X,Y)\sigma &= \nabla^2_X(\omega(Y))\sigma - \nabla^2_Y(\omega(X))\sigma - \omega([X,Y])\sigma \\ &= \nabla^2_X(\omega(Y)\sigma) - \omega(Y)\nabla^2_X\sigma - \nabla^2_Y(\omega(X)\sigma) + \omega(X)\nabla^2_Y\sigma \\ &- \omega([X,Y])\sigma \\ &= \nabla^2_X\nabla^1_Y\sigma - \nabla^2_X\nabla^2_Y\sigma - \omega(Y)\nabla^2_X\sigma \\ &- \nabla^2_Y\nabla^1_X\sigma + \nabla^2_Y\nabla^2_X\sigma + \omega(X)\nabla^2_Y\sigma - \omega([X,Y])\sigma \end{aligned}$$

Now calculate the commutator of $\omega(X)$ and $\omega(Y)$.

$$\begin{aligned} [\omega(X),\omega(Y)]\sigma &= \omega(X)\omega(Y)\sigma - \omega(Y)\omega(X)\sigma \\ &= \omega(X)\nabla^1_Y\sigma - \omega(X)\nabla^2_Y\sigma - \omega(Y)\nabla^1_X\sigma + \omega(Y)\nabla^2_X\sigma \\ &= -\omega(X)\nabla^2_Y\sigma + \omega(Y)\nabla^2_X\sigma \\ &+ \nabla^1_X\nabla^1_Y\sigma - \nabla^2_X\nabla^1_Y\sigma - \nabla^1_Y\nabla^1_X\sigma + \nabla^1_Y\nabla^1_X\sigma \end{aligned}$$

Adding these two equations yields the desired result, because all terms involving ω, except $\omega([X,Y])\sigma$ cancel each other out, as do the mixed terms $\nabla^2_\cdot\nabla^1_\cdot$. ☐

Lemma 2.8. In our situation with ∇ being the covariant derivative of $\mathbb{R}^n$, ∇^B being the Bott connection and ω the connection difference form, we have

$$d_\nabla\omega = -d_{\nabla^B}\omega = -[\omega,\omega].$$

Proof. Let R^B the curvature tensor of the Bott connection and let R^h be the curvature tensor of $\mathbb{R}^n$, restricted to $\mathcal{V}M \times \mathcal{V}M \times \Gamma\mathcal{H}$ (i.e. the domain of R^B). Then, by the preceding lemma, they satisfy $R^h = R^B + d_{\nabla^B}\omega + [\omega,\omega]$. But both connections are flat, so $d_{\nabla^B}\omega = -[\omega,\omega]$. Reversing the roles of R^h and R^B and using $-\omega = \nabla^2 - \nabla^1$, we obtain the other half of the statement. ☐

This allows us to state Gromoll and Walschap's main theorem:

Theorem 2.9 (Main theorem, Gromoll-Walschap, 1997)**.** Let $\pi : \mathbb{R}^{n+k} \to M^n$ be a metric fibration, F the totally geodesic fiber over the soul of M and ω the connection difference form along F. Then ω is closed iff it is Bott-parallel. If this is the case, then

(i) ω induces a Lie algebra homomorphism $\omega : \mathbb{R}^k \to \mathfrak{so}(n)$;

(ii) π is the orbit fibration of the free isometric group action ψ of $\mathbb{R}^k$ on $\mathbb{R}^{n+k} = \mathbb{R}^k \times \mathbb{R}^n$ given by

$$\varphi(v)(u,x) = (u+v, \varphi(v)x), \;\; u,v \in \mathbb{R}^k, \;\; x \in \mathbb{R}^n,$$

where $\varphi : \mathbb{R}^k \to \mathrm{SO}(n)$ is the representation of $\mathbb{R}^k$ induced by ω.

Proof. (i): Clearly, if ω is Bott-parallel, then it is also Bott-closed (i.e. closed w.r.t. d_{∇^B}) and by Lemma 2.8 it is also closed. On the other hand, if X, Y are basic fields along F, and U is vertical, we have $\nabla^v_U(A_X Y) = \nabla^v_U \nabla^v_X Y = \nabla^v_U \nabla_X Y$, since F is totally geodesic and $\nabla^v_U H = -S_H U = 0$ along F for any horizontal H. Since Euclidean space is flat, we get that $\nabla^v_U \nabla_X Y = \nabla^v_X \nabla_U Y + \nabla^v_{[U,X]} Y$. By the same reasoning the last term is zero, so

$$\begin{aligned}
\nabla^v_U(A_X Y) &= \nabla^v_X \nabla_U Y = \nabla^v_X(\nabla^v_U Y + \nabla^h_U Y) \\
&= \nabla^v_X(-S_Y U + \nabla^h_Y U + [U, Y]^h) = \nabla^v_X(-S_Y U - A^*_Y U) \\
&= \nabla^v_X(-S_Y U) - A_X A^*_Y U.
\end{aligned}$$

Using this we have that

$$\begin{aligned}
\langle \nabla^v_U(A_X Y), V\rangle &= \langle \nabla^v_X(-S_Y U), V\rangle - \langle A_X A^*_Y U, V\rangle \\
&= -X\langle S_Y U, V\rangle + \langle S_Y U, \nabla^v_X V\rangle - \langle A^*_Y U, A^*_X V\rangle \\
&= -X\langle S_Y U, V\rangle - \langle \omega(U)Y, \omega(V)X\rangle.
\end{aligned}$$

Let $\alpha = (A_X Y)^\flat$. Then for any U, V vertical and parallel along F

$$\begin{aligned}
d\alpha(U, V) &= \nabla_U \alpha(V) - \nabla_V \alpha(U) \\
&= \langle \nabla^v_U A_X Y, V\rangle - \langle \nabla^v_V A_X Y, U\rangle \\
&= -X\langle S_Y U, V\rangle - \langle \omega(U)Y, \omega(V)X\rangle \\
&\quad + X\langle S_Y V, U\rangle + \langle \omega(V)Y, \omega(U)X\rangle \\
&= \langle \omega(V)Y, \omega(U)X\rangle - \langle \omega(U)Y, \omega(V)X\rangle,
\end{aligned}$$

since S_Y is self-adjoint. Observe that in general

$$\langle [\omega(U), \omega(V)]X, Y\rangle = \langle \omega(U)\omega(V)X, Y\rangle - \langle \omega(V)\omega(U)X, Y\rangle,$$

and

$$\begin{aligned}
\langle \omega(U)\omega(V)X, Y\rangle &= -\langle A^*_{\omega(V)X} U, Y\rangle = -\langle U, A_{\omega(V)X} Y\rangle \\
&= \langle U, A_Y \omega(V)X\rangle = -\langle \omega(U)Y, \omega(V)X\rangle.
\end{aligned}$$

Using Lemma 2.8 we finally obtain

$$\begin{aligned}
d\alpha(U, V) &= \langle \omega(U)\omega(V)X, Y\rangle - \langle \omega(V)\omega(U)X, Y\rangle = \langle [\omega(U), \omega(V)]X, Y\rangle \\
&= -\langle d_\nabla \omega(U, V)X, Y\rangle.
\end{aligned}$$

So if ω is closed, then so is α. But this means $\alpha = df$ for some function f, i.e. $df(U) = \alpha(U) = \langle \alpha, U\rangle = \langle A_X Y, U\rangle$, so $A_X Y$ is a gradient. We know, for any parallel vector field E along a integral curve c of $A_X Y$, that $\langle \frac{\nabla}{\partial t}\dot{c}, E\rangle = \frac{\partial}{\partial t}\langle \nabla f, E\rangle = d\alpha(E)(\frac{\partial}{\partial t}) = 0$, so c must be a geodesic and in particular $A_X Y$ is parallel. As observed before, this is equivalent to ω being Bott-parallel.

(ii): Identify $\mathbb{R}^k$ via parallel translation with its tangent space and view sections of νF as maps $\mathbb{R}^k \to \mathbb{R}^n$. The evaluation of ω at $0 \in \mathbb{R}^k$ then defines a linear map $\omega : \mathbb{R}^k \to \mathfrak{so}(n)$. By Lemma 2.8, ω is a Lie-algebra homomorphism. Let $\varphi : \mathbb{R}^k \to \mathrm{SO}(n)$ be the corresponding group homomorphism. Let X be definied by $X_u = \varphi(u)x$, for $x \in \mathbb{R}^n$. Then X is the basic field with $X_0 = x$:

$$(\nabla_w X)v = \frac{d}{dt}\Big|_{t=0} \varphi(v + tw)x = \frac{d}{dt}\Big|_{t=0} \varphi(tw)\varphi(v)x = \omega(w)X_v.$$

In particular, the fiber $F_{(u,x)}$ of π through a point (u, x) can be described as the set of all $(u + v, X_{u+v})$, as v ranges over $\mathbb{R}^k$. But this is exactly the free action in the statement:

$$\psi(v)(u, x) = (u + v, \varphi(v)x) = (u + v, \varphi(u + v)\varphi(-u)x = (u + v, X_{u+v}),$$

where the last equality follows from the fact that in this case $X_0 = \varphi(-u)x$. □

This concludes the first step. We now restrict our attention to basic vector fields and the vector space $\mathcal{A}$ spanned by all fields $A_X Y$ along the totally geodesic fiber F. Note that by skew-symmetry of the A-tensor, its dimension is at most $n(n-1)/2$. We define the *rank* of the fibration to be the maximum of $\dim \mathcal{A}_p = \dim \{U_p \mid U \in \mathcal{A}\}$. In particular the rank of the fibration is always $\leq k$.

The following lemma shows that $A^*_X A_X$ preserves basic fields for basic X. Then we will use some linear algebra and the fact that $k \leq 3$ to establish that this is also true for $A^*_X A_Y$.

Lemma 2.10. Along leaves, $\langle A_X Y, A_X Z\rangle$ is constant for basic X, Y, Z.

Proof. From [O'N66, Theorem2] we know that for a Riemannian submersion $\pi : M \to B$ the curvature tensor R of M and the horizontal lift R^* of the curvature tensor on B are related via the A-tensor in the following way:

$$\begin{aligned}
&\langle R(X,Y)Z, H\rangle - \langle R^*(X,Y)Z, H\rangle \\
=&\langle A_Y Z, A_X H\rangle + \langle A_Z X, A_Y H\rangle - 2\langle A_X Y, A_Z H\rangle \\
=&\langle A_Y Z, A_X H\rangle - \langle A_X Z, A_Y H\rangle - 2\langle A_X Y, A_Z H\rangle,
\end{aligned}$$

where X, Y, Z, H are horizontal fields on M and we have used the skew-symmetry of the A-tensor. In particular we obtain

$$R^h(X,Y)Z - R^*(X,Y)Z = A^*_X A_Y Z - A^*_Y A_X Z - 2A^*_Z A_X Y.$$

But if M is a space of constant curvature c, as in our case $c = 0$, $R(X,Y)Z = c(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$ and therefore is basic. Hence the right hand side of the equation is also basic. Letting $Z = X$ and using the skew-symmetry of A, the right hand side becomes $-3A_X^*A_XY$. Thus $A_X^*A_XY$ is basic, or equivalently $\langle A_XY, A_XZ\rangle$ is constant along leaves. □

Corollary 2.11. It follows that $A_X^*A_Y + A_Y^*A_X$ also preserves basic fields.

Proof. Straightforward application of the polarization formula for quadratic forms yields for basic X, Y, Z:

$$\begin{aligned}
&A_{X+Y}^*A_{X+Y}Z - A_X^*A_XZ - A_Y^*A_YZ \\
=&A_{X+Y}^*A_XZ + A_{X+Y}^*A_YZ - A_X^*A_XZ - A_Y^*A_YZ \\
=&A_X^*A_XZ + A_Y^*A_XZ + A_X^*A_YZ + A_Y^*A_YZ - A_X^*A_XZ - A_Y^*A_YZ \\
=&A_X^*A_YZ + A_Y^*A_XZ
\end{aligned}$$

The left hand side is basic by the previous lemma. Hence the right hand side is also basic. □

Lemma 2.12. For X, Y, Z, W being basic vector fields, $\langle A_XY, A_ZW\rangle$ is constant along F^k.

Proof. First we deal with the substantial case. Clearly we do not need to consider the case $k = 1$. Note that if X, Y, Z, W are not linearly independent, we can use the skew-symmetry of the A-tensor and immediately obtain the result using Lemma 2.10. In particular, we may assume $n \geq 4$.

Consider a subspace H of basic fields along F^k such that $\dim H = m + 1 \leq 4$ and for some $X_0 \in H$ the vertical space is spanned by $\{A_{X_0}Y \mid Y \in H\}$ at some point in the fiber. By Lemma 2.10 the vertical distribution is spanned by $\{A_{X_0}Y \mid Y \in H\}$ along all of F^k. The case $m = 2$ can only occur if $k = 2$ and for $n \geq 4$ we may assume $m = 3$ by enlarging H if necessary. So we only have to deal with this case if $k = 2$ and $n = 2$.

Now the claim holds for all $X, Y, Z, W \in H$: Let X_0 be as above and choose $X_l \in H$ for $1 \leq l \leq k$ such that

(i) the X_l are independent for $0 \leq l \leq m$, and

(ii) the $A_{X_0}X_l$ are orthonormal for $1 \leq l \leq k$.

We only have to show that $\langle A_{X_i}X_j, A_{X_0}X_l\rangle$ is constant for $0 \leq i < j \leq m, 1 \leq l \leq k$, since we can use the skew-symmetry of A to obtain the other cases. But for $i = 0, i = l$ or $j = l$ we already know by Lemma 2.10 that this is true. For the other cases we use that $\|A_{X_i}X_j\|$ is constant along fibers. Observe that, by (ii), we have

$$\|A_{X_i}X_j\|^2 = \sum_{l=1}^{k}\langle A_{X_i}X_j, A_{X_0}X_l\rangle^2.$$

Now we use $k \leq 3$; careful observation of the indices and using the skew-symmetry of A shows that the right hand side consists only of constant terms by Lemma 2.10, except for $\langle A_{X_i}X_j, A_{X_0}X_l\rangle$.

If for any basic X, Y, Z, W, after possibly reordering, we have $A_W X$, $A_W Y$ and $A_W Z$ span the vertical distribution, set $H = \operatorname{span}\{X, Y, Z, W\}$. However if this does not hold, we use the assumption that the fibration is substantial and extend X, Y, Z, W to a spanning set of such an H of dimension $m + 1$. This concludes the substantial case.

Now assume that $\operatorname{rank} A_W < k$. Then $\dim\ker A_w > n - r \geq n - 3$. If X, Y, Z are linearly dependent, say $Z = aX + bY$ we have by skew-symmetry of A

$$\langle A_X Y, A_Z W\rangle = \langle A_X Y, A_{aX+bY}W\rangle = a\langle A_X Y, A_X W\rangle + b\langle A_X Y, A_Y W\rangle,$$

but this is constant as $A_X^* A_X$ preserves basic fields. So let X, Y, Z be linearly independent. Then $\ker A_W \cap \operatorname{span}\{X, Y, Z\} \neq \emptyset$. Let therefore $aX + bY + cZ \in \ker A_W$, with say, $a \neq 0$. Then

$$a\langle A_W X, A_Y Z\rangle + b\langle A_W Y, A_Y Z\rangle + c\langle A_W Z, A_Y Z\rangle = 0,$$

where the last two summands are constant, because they only involve three basic vectors each. Hence

$$\langle A_W X, A_Y Z\rangle = \langle A_X W, A_Z Y\rangle = \langle A_Z^* A_X W, Y\rangle$$

is constant, i.e. $A_Z^* A_X W$ is basic. We know that $A_Z^* A_X + A_X^* A_Z$ preserves basic fields, so $\langle A_Z^* A_X W + A_X^* A_Z W, Y\rangle$ is constant, too. But then also $\langle A_X^* A_Z W, Y\rangle = \langle A_Z W, A_X Y\rangle$ must be constant. □

Corollary 2.13. For $\mathcal{F}$ weakly substantial, homogeneity follows.

Proof. We have to show that $\langle A_X Y, U\rangle$ constant for parallel U. But $U \in \mathcal{A}$ by assumption, so $U = A_Z W$ for basic Z, W and $\langle A_X Y, A_Z W\rangle$ is constant and we may apply Theorem 2.9. □

Now the dimension of $\mathcal{A}$ coincides with the rank of the foliation and $\langle A_X Y, A_Z W\rangle$ is constant along F for basic X, Y, Z, W. In other words $\langle A_X^* A_Z W, Y\rangle = -\langle \omega(A_Z W)X, Y\rangle$, i.e. $\omega(V)$ preserves basic fields for $V \in \mathcal{A}$. This allows us to show that $\mathcal{A}$ is a Lie algebra.

Lemma 2.14 ($\mathcal{A}$ is a Lie algebra)**.** $(\mathcal{A}, \{\cdot,\cdot\})$ is a Lie algebra isomorphic to a subalgebra of $\mathfrak{so}(n)$, where $\{U, V\}$ is the orthogonal projection of $[U, V]$ onto $\mathcal{A}$.

Proof. As observed, $\omega(V)$ preserves basic fields for $V \in \mathcal{A}$. Let X, Y be basic extensions of x and y and set $\langle -\omega(V)x, y\rangle = -\langle \omega(V)X, Y\rangle$. We obtain a linear isomorphism between $\mathcal{A}$ and $\omega(\mathcal{A}) \subset \mathfrak{so}(n)$. It is clearly onto, and to see that it is injective, let $V \in \ker \omega$. Then for all x, y and corresponding basic extensions X, Y

$$0 = \langle \omega(V)x, y\rangle = \langle -A_X^* V, Y\rangle = -\langle V, A_X Y\rangle.$$

But $A_X Y \in \mathcal{A}$ by definition of $\mathcal{A}$, so V must be the zero vector field. Now consider the lie bracket $[\cdot, \cdot]$ of $\mathfrak{so}(n)$. Then for basic X and $U, V \in \mathcal{A}$

$$[\omega(U), \omega(V)]X = \omega(U)\omega(V)X - \omega(V)\omega(U)X = \nabla^h_{\nabla^h_X V} U - \nabla^h_{\nabla^h_X U} V.$$

Since $\omega(V)$ preserves basic fields, $[U, \nabla^h_X V]$ is vertical and it follows that

$$[\omega(U), \omega(V)]X = \nabla^h_U \nabla^h_X V - \nabla^h_{\nabla^h_X U} V = \nabla^h_U \omega(V)X - \omega(V)(\nabla^h_X U)$$

Note that because F^k is totally geodesic any field of the form $\nabla_U X$ with U vertical and X horizontal is horizontal along F^k:

$$\nabla_U X = \nabla^h_U X + \nabla^v_U X = \nabla^h_U X - S_X U = \nabla^h_U X$$

Now we see that

$$\begin{aligned}\nabla_U(\omega(V))X &= \nabla_U \omega(V)X - \omega(V)(\nabla_U X)\\ &= \nabla^h_U \omega(V)X - \omega(V)(\nabla_U X) = [\omega(U), \omega(V)]X.\end{aligned}$$

We know from Lemma 2.8 that the exterior derivative of ω w.r.t. ∇ is $d\omega = -[\omega, \omega]$. But also:

$$\begin{aligned}d\omega(U, V) &= \nabla_U \omega(V) - \nabla_V \omega(U) - \omega([U, V])\\ &= [\omega(V), \omega(U)] - [\omega(U), \omega(V)] - \omega([U, V])\\ &= -2[\omega(U), \omega(V)] - \omega([U, V]),\end{aligned}$$

and consequently

$$\begin{aligned}-[\omega, \omega](U, V) &= -2[\omega, \omega](U, V) - \omega([U, V]).\\ &\Leftrightarrow [\omega(U), \omega(V)] = -\omega([U, V]).\end{aligned}$$

Hence ω is actually an isomorphism of Lie algebras and we can define a bracket on $\mathcal{A}$ via

$$U, V := (-\omega)^{-1}([\omega(U), \omega(V)]) = \omega^{-1}(\omega([U, V]) = [U, V] - [U, V]^{\ker \omega},$$

where $\cdot^{\ker \omega}$ denotes the projection on the kernel of ω. Notice that $U \in \ker \omega$, iff $\omega(U)X = -A_X^* U = 0$ for all horizontal X. But this means that for any horizontal Y $\langle A_X^* U, Y\rangle = \langle A_X Y, U\rangle = 0$, so $\ker \omega = \mathcal{A}^\perp$ and the claim follows. □

Recall that for the application of Theorem 2.9 we need that ω is exact, or equivalently that $[\omega, \omega] = 0$. For this we need a better understanding of the Lie bracket $\{\cdot, \cdot\}$ of $\mathcal{A}$. It will follow from the following two lemmata, that $\{\cdot, \cdot\}$ is just the Lie bracket of $\mathcal{A}$, i.e. of $\mathbb{R}^r$, where r is the rank of the fibration.

Lemma 2.15 ($\mathcal{A}^\perp$ defines a metric foliation)**.** The distribution $\ker \omega = \mathcal{A}^\perp$ generates a Riemannian foliation $\mathcal{F}$ on F.

Proof. Let $T_1, T_2 \in \ker \omega$. By Lemma 2.14, $\omega([T_1, T_2]) = [\omega(T_1), \omega(T_2)] = 0$. Hence $\ker \omega$ is integrable by Frobenius' Theorem and generates a foliation $\mathcal{F}$. Note that by the same argument for $T \in \ker \omega$ we have $\omega([T, A_X Y]) = -d\omega(T, A_X Y) = [\omega(T), \omega(A_X Y)] = 0$. In particular, we have for X, Y, Z, W basic

$$\langle [T, A_X Y], A_Z W\rangle = \langle A_Z^*[T, A_X Y], W\rangle = -\langle \omega([T, A_X Y]), W\rangle = 0 \qquad (*)$$

To see that the foliation is Riemannian, we show that the corresponding Bott connection $[T, X]^{\mathcal{A}}$ is metric, where X is horizontal and $\cdot^{\mathcal{A}}$ denotes the projection onto $\mathcal{A}$. For this calculation, we employ the Einstein summation convention. Let X_i, Y_i be basic fields such that $A_i := A_{X_i} Y_i$ is an orthonormal basis of $\mathcal{A}$ and let $X = \varphi^i A_i$, $Y = \psi^i A_i$ be horizontal vectorfields (w.r.t. $\mathcal{F}$). By $(*)$, we now have $\langle [T, \varphi^i A_i], A_j\rangle = \langle T(\varphi^i)A_i + \varphi^i[T, A_i], A_j\rangle = \langle T(\varphi_i)A_i, A_j\rangle$.

$$\begin{aligned}
T\langle X, Y\rangle &= T\langle \varphi^i A_i, \psi^i A_i\rangle = \langle \nabla_T \varphi^i A_i, \psi^i A_i\rangle + \langle \varphi^i A_i, \nabla_T \psi^i A_i\rangle \\
&= \psi^j \langle \nabla_T \varphi^i A_i, A_j\rangle + \varphi^j \langle A_j, \nabla_T \psi^i A_i\rangle \\
&= \psi^j \langle T(\varphi^i)A_i + \varphi^i \nabla_T A_i, A_j\rangle + \varphi^j \langle A_j, T(\psi^i)A_i + \psi^i \nabla_T A_i\rangle \\
&= \psi^j \langle T(\varphi^i)A_i, A_j\rangle + \varphi^j \langle A_j, T(\psi^i)A_i\rangle \\
&\quad + \varphi^j \varphi^i \langle \nabla_T A_i, A_j\rangle + \varphi^j \psi^i \langle A_j, \nabla_T A_i\rangle \\
(*) &= \psi^j \langle [T, \varphi^i A_i], A_j\rangle + \varphi^j \langle A_j, [T, \psi^i A_i]\rangle + \varphi^i \psi^j T\langle A_i, A_j\rangle \\
&= \langle [T, \varphi^i A_i], \psi^i A_i\rangle + \langle \varphi^i A_i, [T, \psi^i A_i]\rangle \\
&= \langle [T, X]^{\mathcal{A}}, Y\rangle + \langle X, [T, Y]^{\mathcal{A}}\rangle.
\end{aligned}$$

□

Lemma 2.16 (The bracket of $\mathcal{A}$)**.** The bracket $\{\cdot, \cdot\}$ of $\mathcal{A}$ coincides with the ordinary Lie bracket $[\cdot, \cdot]$.

Proof. Note that, it holds that $\langle \nabla_U V, W\rangle$ for $U, V, W \in \mathcal{A}$ is constant, because for basic X, Y $\langle \nabla_U V, A_X Y\rangle = \langle A_X^* \nabla_U^{\mathcal{A}} V, Y\rangle = -\langle \omega(\nabla_U^{\mathcal{A}} V)X, Y\rangle$, while $\omega(U)$ preserves basic fields for $U \in \mathcal{A}$.

We now examine $\mathcal{F}$ as defined in Lemma 2.15. If κ denotes the mean curvature form of $\mathcal{F}$, we have for an orthonormal base U_i of $\mathcal{A}$ and an orthonormal base V_i of $\mathcal{A}^\perp$,

$$\begin{aligned}\kappa(A_X Y) &= -\sum_i \langle V_i, \nabla_{V_i} A_X Y\rangle \\ &\qquad = -\operatorname{div} A_X Y + \sum_i \langle U_i, \nabla_{U_i} A_X Y\rangle = \sum_i \langle U_i, \nabla_{U_i} A_X Y\rangle,\end{aligned}$$

where div is the divergence of the fiber F and hence $\operatorname{div} A_X Y = 0$. It follows that κ is constant on each element of a basic spanning set for $\mathcal{A}$ and consequently $d\kappa = 0$. Hence $\kappa = df$ for some function f and $\kappa^\sharp = \nabla f$ is a gradient.

Let c be an integral curve of ∇f; c is necessarily a geodesic (as seen in the proof of Theorem 2.9). Then $\kappa(\dot c) = \kappa(\nabla f) = \langle \kappa^\sharp, \nabla f\rangle = \langle \kappa^\sharp, \kappa^\sharp\rangle = \|\kappa\|^2$. But $\kappa(\dot c) = \langle \dot c, \dot c\rangle = \|\dot c\|^2$, so $\|\kappa\|^2$ is constant.

We have seen in the proof of Lemma 1.34 that $\|S_{\dot c} U\|^2$ goes to zero, so we actually have $\kappa \equiv 0$, and hence the foliation is by minimal leaves.

Hence Lemma 1.34 applies, thus for $X, Y \in \mathcal{A}$, $X, Y = [X,Y]^{\mathcal{A}} = [X,Y] - [X,Y]^{\mathcal{A}^\perp} = [X,Y] - 2A_X Y = [X,Y]$. □

Proof of Theorem 2.6. Now everything comes together. By Lemma 2.14 we have $(\mathcal{A}, [\cdot,\cdot]^{\mathcal{A}})$ is a Lie algebra, where $\cdot^{\mathcal{A}}$ denotes the orthogonal projection onto $\mathcal{A}$. Now Lemma 2.15 tells us, that $\mathcal{A}^\perp = \ker\omega$ generates a Riemannian foliation $\mathcal{F}$ of the fiber F^k where the $A_X Y$ are basic fields. As noted above, this foliation is minimal, because $\|S_{\dot c(t)}\| \to 0$ for $t \to \infty$, as seen in the proof of Lemma 1.34. But this means that it is also flat, i.e. $[U,V]^{\mathcal{A}^\perp} = 0$ for $U, V \in \mathcal{A}$. Hence $(\mathcal{A}, [\cdot,\cdot]^{\mathcal{A}}) = (\mathcal{A}, [\cdot,\cdot])$ is isomorphic to a Lie subalgebra of $\mathfrak{so}(n)$.

Let e_i denote an orthonormal basis of $-\omega(\mathcal{A})$. By calculating the structure constants $c_{ijk} = \langle [e_i, e_j], e_k\rangle_{\mathfrak{so}(n)}$, we find that $e_l = -\omega(f_l)$ for some $f_l \in \mathcal{A}$. But $-\omega$ is an isomorphism of Lie algebras and preserves brackets, and because the bracket of $\mathcal{A}$ is zero, it follows that $[e_i, e_j]$ must also be zero. Hence the structure constants c_{ijk} are also zero. By [Mil76, Lemma 1.1], the curvature of $-\omega(\mathcal{A})$ may be expressed in terms of these structure constants and is consequently flat. Then, by [Mil76, Theorem 1.5], $-\omega(\mathcal{A})$ now splits as an orthogonal direct sum of a commutative subalgebra and a commutative ideal. But with $\mathfrak{so}(n)$ being semisimple, the latter has to be zero, hence $-\omega(\mathcal{A})$ is Abelian. In particular $d\omega = [\omega, \omega] = 0$ and Theorem 2.9 applies. □

3 A theorem on existence of local foliations

We have seen that whenever we have a foliation, its normal bundle is flat w.r.t. the Bott connection. If X is a basic field along a leaf L and c a curve in L, Y the parallel transport of $\dot{c}(0)$ along an integral curve of X emanating from $c(0)$, then $\nabla^h_{\dot{c}(0)} X = \nabla^h_Y X = \nabla^h_X Y + [Y, X]^h = 0$, so $\nabla_{\dot{c}} X$ is tangential to L. This in turn is just the canoncial connection on the normal bundle of a submanifold, where a normal field is parallel if its covariant derivative in directions of the submanifold is tangential.

The following theorem shows that the converse of this observation holds in Euclidean space.

Theorem 3.1. Let $(N^{n-k}, g) \subset (\mathbb{R}^n, \langle\ ,\ \rangle)$ be a Riemannian submanifold with flat normal bundle. Then N locally defines a Riemannian foliation of $\mathbb{R}^n$. Furthermore the basic vector fields are exactly those parallel and normal to N.

Proof. Since νN is flat, there are locally parallel sections of the normal bundle. Let $p \in N$, $U \subset \mathbb{R}^n$ be an open neighbourhood of p small enough, such that $\nu(U \cap N)$ admits parallel sections and for a parallel section X that $\exp X(N) \cap U$ yields a diffeomorphic copy of $N \cap U$. Clearly this defines a foliation, with vertical vectors being those tangent to copies of $N \cap U$. Let X be a parallel section, let $N_s := \exp sX(N) \cap U$ for all s where the intersection is not empty. Let $q_s = \exp_m sX$ for some $m \in N$. We then may identify the tangent spaces $T_m N$ and $T_{q_s} N_s$ via parallel translation along the geodesic $s \mapsto \exp_m sX$.

Now we wish to see that if $e \in T_m\mathbb{R}^n$ is orthogonal to N, then the parallel transport E of e along $s \mapsto \exp_m sX$ is again orthogonal to N_s. Let $v \in T_{q_s} N_s$, γ be a curve with $\gamma(0) = q_s$, $\dot{\gamma}(0) = v$. Note that $\gamma(t) = \exp_{c(t)} sX$ for some curve c in N with $c(0) = m$. Then $\exp_{c(t)} sX(c(t))$ is a variation through geodesics, with variational field J, the Jacobi field with initial conditions $J(0) = \dot{c}(0)$ and

$$\begin{aligned} J'(0) &= \frac{\nabla}{\partial s}\bigg|_{s=0} \frac{\partial}{\partial t}\bigg|_{t=0} \exp_{c(t)} sX(c(t)) \\ &= \frac{\nabla}{\partial t}\bigg|_{t=0} \frac{\partial}{\partial s}\bigg|_{s=0} \exp_{c(t)} sX(c(t)) = \frac{\nabla}{\partial t}\bigg|_{t=0} X(c(t)) = 0, \end{aligned}$$

because X is parallel. Furthermore, we know in Euclidean space, $J'' = 0$. It follows that J is the parallel transport of $\dot{c}(0)$ along $s \mapsto \exp_m sX$, so $J(1) = v$. By assumption $\langle J(0), E(0)\rangle = \langle \dot{c}(0), e\rangle = 0$. Furthermore,

$$\langle J, E\rangle' = \langle J', E\rangle + \langle J, E'\rangle = 0,$$

since both fields are parallel. Consequently,

$$\langle J, E\rangle'' = 2\langle J', E'\rangle + \langle J'', E\rangle + \langle J, E''\rangle = 0.$$

Hence $\langle J, E\rangle \equiv 0$.

Now we can show that this foliation is Riemannian: Let U be vertical, X, Y be horizontal, $p = \exp Z(m) \in N^Z$ with $N^Z = \exp(Z(N)) \cap U$ for some parallel section Z. Then $U(p) = \dot{c}(0)$ for some curve in N^Z. Let e_i be an orthonormal base of the horizontal space at m, and extend it via parallel transport in Euclidean space to an parallel orthonormal frame E_i. In particular, we have $\nabla_{E_i} E_j = 0$. As observed above, then the $E_i(p)$ are still horizontal. Using the Einstein summation convention, write $X = \varphi^i E_i$ and $Y = \psi^i E_i$. Then

$$U\langle X, Y\rangle = \langle \nabla_U X, Y\rangle + \langle X, \nabla_U Y\rangle = \langle \nabla_X U + [U, X], Y\rangle + \langle X, \nabla_Y U + [U, Y]\rangle.$$

By Lemma 1.37, we are done if we can show that $\nabla^h_X U = 0$. First observe that

$$\langle \nabla_X U, Y\rangle = X\langle U, Y\rangle - \langle U, \nabla_X Y\rangle = -\langle U, \nabla_X Y\rangle.$$

But we have

$$\begin{aligned} \nabla_X Y &= \nabla_{\varphi E_i} \psi^j E_j = \varphi^i \nabla_{E_i} \psi^j E_j \\ &= \varphi^i \psi^j \nabla_{E_i} E_j + \varphi^i E_i(\psi^j) E_j, \end{aligned}$$

and by construction of the E_i, the first summand vanishes. This means that $\nabla_X Y$ is horizontal, hence $\langle \nabla_X U, Y\rangle = -\langle U, \nabla_X Y\rangle = 0$, establishing the claim. $\square$

4 A small survey on generalizations

We conclude this thesis by surveying on some of the possible generalizations of the main theorem Theorem 2.6.

4.1 Homogeneity in arbitrary codimension

Probably the most obvious generalization is to check whether the result holds for a Riemannian submersion $\pi : \mathbb{R}^{n+k} \to M^n$ with $k > 3$. Gromoll and Walschap published a proof in [GW01], however, according to [FGLT00] Stefan Weil found an error in it, which in turn was communicated to the author by Marco Radeschi. However, the proof still works whenever the fibration is substantial. We will give a short outline of the strategy of the proof.

As in the case for $k \leq 3$, one obtains a totally geodesic fiber F^k over the soul of M^n. Again, the goal is to apply Theorem 2.9, i.e. to check that the connection difference form $\omega = -A^*$ is basic. However, the arguments of Lemma 2.12 clearly do not generalize appropriately, since we relied heavily on the fact that $k \leq 3$. In the general case, one considers, an parallel orthonormal base $E_1, \dots, E_k$ of the vertical distribution along the fiber F^k and extends it via holonomy diffeomorphism radially to vector fields $U_1, \dots, U_k$ on all of $\mathbb{R}^{n+k}$. In particular, along horizontal geodesics c, $U_i \circ c$ is just the holonomy field with initial conditions $J(0) = E_i$, $J'(0) = -A_{\dot{c}}E_i$, since F^k is totally geodesic.

Recalling that the mean curvature form is basic and exact, let f be the function with $df = \kappa$ and $f|_{F^k} \equiv 1$, and be τ the vertical volume form restricted to vertical fields. Then define the *holonomy form* $\eta := e^{-f}\tau$. Now consider the k-blades $U_1 \wedge \cdots \wedge U_k$. It turns out that these are dual to the holonomy form. This allows one to establish that they are holonomy invariant in the sense that the wedge is independent of the chosen horizontal path.

Using the usual identifications of $\mathbb{R}^{n+k}$ with its tangent space, one treats $\eta^\sharp = U_1 \wedge \cdots \wedge U_k$ as a map $\mathbb{R}^{n+k} \to \bigwedge_k \mathbb{R}^{n+k}$. Since the U_i are holonomy fields, they are linear along horizontal geodesics emanating from the totally geodesic fiber. This in turn implies that $\eta^\sharp$ is polynomial of degree at most k along horizontal subspaces $\mathbb{R}^n \times \{v\}, v \in F^k$ in the sense that the component functions $\varphi_i : \mathbb{R}^{n+k} \to \mathbb{R}$ are polynomial of degree at most k. Now the holonomy invariance allows one to show that $\eta^\sharp$ is indeed polynomial along any affine horizontal subspace.

The form of holonomy fields in Euclidean space shows that the vertical space at infinity is spanned by $(\ker + \operatorname{im})A^*$, and equivalently, the horizontal space at infinity is spanned by $(\ker + \operatorname{im})A$. A continuity argument yields that $\eta^\sharp$ is polynomial along any affine plane through a point $(0, v) \in F^k$ spanned by a horizontal vector x and a vertical $u \in \operatorname{im} A_x$. Now, $\nabla_x \eta^\sharp$

restricted to a line $\gamma_u(t) := (0, v + tu)$ is given by

$$\nabla_x \eta^\sharp = -\sum_i E_1 \wedge \cdots \wedge A_x^* E_i \wedge \cdots \wedge E_k.$$

The E_j are parallel and because $\|A_X Y\|$ is constant along fibers, $A_x^* E_i$ is bounded in norm as well. The derivative of a polynomial map is also polynomial and a bounded polynomial must be constant. It follows that each $A_x^* E_i$ is parallel along γ_u, or equivalently, that $(A_x y \circ \gamma_u)' \equiv 0$ for $u \in \operatorname{im} A_x$, x, y parallel sections of νF^k. Hence, $A_X Y$ is parallel along F in directions $u \in \operatorname{im} A_x$, so Theorem 2.9 applies in the case of a substantial foliation.

For the non-substantial case Gromoll and Walschap gave a variational argument, however, the initial conditions are *a priori* only fulfilled at $0 \in \mathcal{F}^k$, but are needed along the whole fiber, so one is not allowed to split off $\mathcal{A}^\perp$ in a way similar to Lemma 2.15.

4.2 Another approach to show homogeneity

The proofs of Gromoll and Walschap first established that ω is basic and defines a Lie algebra homomorphism. The corresponding Lie group homomorphism $\varphi : \mathbb{R}^k \to \mathrm{SO}(n)$ yields that the leaves of the foliation are given by

$$\left\{ (v, \varphi(v)x \mid x \in \mathbb{R}^n, v \in \mathbb{R}^k \right\}.$$

Wilking suggested that one could try to reverse this approach. Since the soul construction yields a totally geodesic fiber, one can obtain a map $\varphi : \mathbb{R}^k \to \mathrm{SO}(n)$ in another fashion: Let $E_i = (v, x_i)$ denote the basic field which projects to the same vector field as $(0, e_i) \in T_0\mathbb{R}^{n+k}$ for an orthonormal basis of $\{0\} \times \mathbb{R}^n$. Then let φ be defined via $\varphi(v)e_i = x_i$. Now the leaves are again of the form given above, but one does not know yet that φ is a homomorphism of Lie groups. One might be able to exploit the fact that the leaves are equidistant.

As we learned that the approach of Gromoll and Walschap does not work in all cases, Wilking also suggested that one might be able to rule out a class of counterexamples, namely those equidistant fibrations arising from arbitrary group actions, and not only Abelian ones, by showing that these must be at least orbit equivalent to an equidistant fibration arising from an Abelian Lie group action. In this case, the existence of the totally geodesic fiber yields a similar map, again.

4.3 Foliations

So far we have only dealt with the case where we have a Riemannian submersion $\pi : \mathbb{R}^{n+k} \to M^n$. But what is the situation if we are only given a k-dimensional foliation $\mathcal{F}$ of $\mathbb{R}^{n+k}$? Gromoll and Walschap showed in

[GW97] that for $k \leq 2$, $\mathcal{F}$ actually is a fibration, hence the classification result applies.

In fact, Florit, Goertsches, Lytchak and Töben showed in [FGLT00] that any foliation of Euclidean space is actually given by a fibration and so they reduced the classifications of foliations to the one of fibrations. In particular, using the results presented in this thesis, one can show that foliations of codimension $k \leq 3$ are homogeneous, as are substantial foliations in arbitrary codimension.

4.4 Singular equidistant foliations

When talking about foliations, we always assume that the leaves share the same dimension. While this restriction seems reasonable, dropping it yields many more interesting examples, like the Euclidean plane foliated by concentric circles of different radii, with the circle of radius 0 being a singular leaf of dimension 0. Boltner studied these in [Bol07], while keeping the assumption that the leaves are equidistant. In this case the map $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}/\mathcal{F}$ is not a Riemannian submersion, but $\mathbb{R}^{n+k}/\mathcal{F}$ has the structure of an Alexandrov space and π is a submetry. Boltner has adapted the soul construction to this setting (his approach also works in the case of a Riemannian submersion), and shows how one can obtain new inhomogeneous examples from existing ones, as these foliations do not necessarily need to be homogeneous.

4.5 Curvature

One of the main topics in differential geometry is that of curvature, and one is interested in the behaviour of structures in different curvature settings. So a question that naturally arises is whether one can obtain homogeneity in non-flat spaces. Indeed, the groundwork for the study of Riemannian foliation was laid down by Gromoll and Grove in [GG85], where the authors study one dimensional foliations in spaces of constant curvature K. They obtain that for $K \geq 0$, a foliation must be either homogeneous or flat, and since the latter can only occur for $K = 0$, foliations of spheres are always congruent to Hopf fibrations $\mathbb{S}^{2n+1} \to \mathbb{C}P^n$. For $K < 0$ no such result holds, even in the one-dimensional case: One can consider the foliation generated by exponentiating parallel sections of the normal bundle of a line without focal points, hence this foliation is global. By a version of Theorem 3.1 this foliation is metric, and in negative curvature one can disturb the line quite arbitrarily, without introducing focal points.

Later, in [GG88], Gromoll and Grove extend the result for space forms of positive curvature to foliations of codimension $k \leq 3$. There are topological restrictions on what fibrations a sphere can admit: Any fibration of a homotopy sphere $\mathbb{S}^n$ must have fibers $\mathbb{S}^1$, $\mathbb{S}^3$ or $\mathbb{S}^7$, but the latter can only occur in the case $n = 15$. So except for this case, they classified the fibrations of spheres, which are always given by Hopf fibrations. As already mentioned, their proof also relies heavily on linear algebra and the assumption that $k \leq 3$.

Only recently the remaining case was classified by Lytchak and Wilking in [LW13]. Their approach is quite different, employs a lot of topological machinery and finally yields the following classification result:

Theorem 4.1 (Lytchak-Wilking, 2013)**.** Let $\mathcal{F}$ be a Riemannian foliation on a round sphere $\mathbb{S}^n$ with leaf dimension $0 < k < n$. Then, up to isometric congruence, either $\mathcal{F}$ is given by the orbits of an isometric action of $\mathbb{R}$ or $\mathbb{S}^3$ with discrete isotropy groups or it is the Hopf fibration of $\mathbb{S}^{15} \to \mathbb{S}^8(1/2)$ with fiber $\mathbb{S}^7$.

Acknowledgements

I want to express my gratitude towards Prof. Dr. Burkhard Wilking for his supervision and for providing me with the opportunity to write on the subject of Riemannian submersions, as well as the interesting and motivating discussions. Furthermore I want to thank Marco Radeschi for answering my questions regarding the error in the proof of [GW01], and Martin Kerin and Ricardo Mendes for helping me to understand Lie algebras and the argument leading to the result in small codimension. Also I want to thank Martin Maiwald for proofreading this thesis and many helpful suggestions. All remaining errors are of course my own. Finally, I want to thank my grandmother for her continued moral support throughout my studies.

References

[Bol07] Christian Boltner. *On the Structure of Equidistant Foliations of Euclidean Space*. PhD thesis, Universität Augsburg, 2007.

[CG72] Jeff Cheeger and Detlef Gromoll. On the structure of complete manifolds of nonnegative curvature. *Ann. of Math. (2)*, 96:413–443, 1972.

[FGLT00] Luis Florit, Oliver Goertsches, Alexander Lytchak, and Dirk Töben. Riemannian foliations on contractible manifolds, 2000, arXiv:1309.1989v1.

[GG85] Detlef Gromoll and Karsten Grove. One-dimensional metric foliations in constant curvature spaces. In *Differential geometry and complex analysis*, pages 165–168. Springer, Berlin, 1985.

[GG88] Detlef Gromoll and Karsten Grove. The low-dimensional metric foliations of Euclidean spheres. *J. Differential Geom.*, 28(1):143–156, 1988.

[GP97] Luis Guijarro and Peter Petersen. Rigidity in non-negative curvature. *Ann. Sci. École Norm. Sup. (4)*, 30(5):595–603, 1997.

[GW97] Detlef Gromoll and Gerard Walschap. Metric fibrations in euclidean space. *ASIAN JOURNAL OF MATHEMATICS*, 1:716–728, 1997.

[GW01] Detlef Gromoll and Gerard Walschap. The metric fibrations of Euclidean space. *J. Differential Geom.*, 57(2):233–238, 2001.

[GW09] Detlef Gromoll and Gerard Walschap. *Metric foliations and curvature*, volume 268 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 2009.

[Lee13] John M. Lee. *Introduction to smooth manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2013.

[LW13] Alexander Lytchak and Burkhard Wilking. Riemannian foliations of spheres, 2013, arXiv:1309.7884v1.

[Mil76] John Milnor. Curvatures of left invariant metrics on Lie groups. *Advances in Math.*, 21(3):293–329, 1976.

[O'N66] Barrett O'Neill. The fundamental equations of a submersion. *Michigan Math. J.*, 13:459–469, 1966.

[O'N67] Barrett O'Neill. Submersions and geodesics. *Duke Mathematical Journal*, 34(2):363–373, 06 1967.

Plagiatserklärung

Hiermit versichere ich, dass die vorliegende Arbeit über Riemannian Submersions of Euclidean Space selbstständig verfasst worden ist, dass keine anderen Quellen und Hilfsmittel als die angegebenen benutzt worden sind und dass die Stellen der Arbeit, die anderen Werken — auch elektronischen Medien — dem Wortlaut oder Sinn nach entnommen wurden, auf jeden Fall unter Angabe der Quelle als Entlehnung kenntlich gemacht worden sind.

(Ort, Datum, Unterschrift)

Ich erkläre mich mit einem Abgleich der Arbeit mit anderen Texten zur Auffindung von Übereinstimmungen, sowie mit einer zu diesem Zweck vorzunehmenden Speicherung der Arbeit in eine Datenbank einverstanden.

(Ort, Datum, Unterschrift)